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2 + 1 DIMENSIONAL GRAVITY AS AN EXACTLY SOLUBLE SYSTEM

Edward WITTEN*

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

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By disentangling the hamiltonian constraint equations, 2 + 1 dimensional gravity (with or without a cosmological constant) is shown to be exactly soluble at the classical and quantum levels. Indeed, it is closely related to Yang–Mills theory with purely the Chern–Simons action, which recently has turned out to define a soluble quantum field theory. 2 + 1 dimensional gravity has a straightforward renormalizable perturbation expansion, with vanishing beta function. 2 + 1 dimensional quantum gravity may provide a testing ground for understanding the role of classical singularities in quantum mechanics, may be related to the discrete series of Virasoro representations in 1 + 1 dimensions, and may be a useful tool in studying three-dimensional geometry.

1. Introduction

There are two bits of standard folklore about general relativity in 2 + 1 dimensions*. One piece of folklore holds that this system is "trivial", on the grounds that there are no gravitational waves in this low dimension. ("Nontriviality" can be achieved by including matter fields, but we will not do that in this paper.) The other bit of folklore holds that general relativity is inconsistent in 2 + 1 dimensions, since it is supposedly unrenormalizable.

Clearly, there is a certain amount of tension between these two pieces of folklore. It is surprising to be told that a "trivial" system suffers from intractable infinities. Actually, if we probe a little bit deeper, there is a very definite contradiction between the claim that general relativity is trivial in 2 + 1 dimensions and the claim that it is unrenormalizable. The contradiction appears if we contemplate the meaning of "quantization". What it means to quantize a theory is to construct the classical phase space, define Poisson brackets on this space, and then interpret the functions on phase space as quantum mechanical operators. When one says that general relativity is "trivial" in 2 + 1 dimensions one means that the classical phase spaces that arise with reasonable boundary conditions are finite dimensional (as opposed to infinite-dimensional phase spaces which are said to be "non-trivial").

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* For various investigations of this problem from different viewpoints, see refs. [1–8].

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We cannot possibly run into problems of unrenormalizability in trying to quantize a finite-dimensional phase space. Depending on its topology, a finite-dimensional phase space might be unquantizable, but this is not the kind of problem that is envisaged by the bit of folklore which says that 2 + 1 dimensional gravity does not make sense.

To make these considerations a little bit more precise, let us analyze the possible phase spaces (depending on boundary conditions) in 2 + 1 dimensional gravity. First of all, what is "classical phase space"? Phase space is often defined as the space of all values of q_i and $\dot{q}_i$ (the positions and momenta) at time zero, subject to possible constraint equations in the case of a gauge theory. This definition is not manifestly covariant and may, in general, lead to a lengthy analysis of constraint equations – though in sect. 2 we will see that the constraints can be neatly untangled in the case of 2 + 1 dimensional gravity. There is another definition of classical phase space that is manifestly covariant: classical phase space is the space of all solutions of the classical equations, modulo gauge transformations. In fact the role of specifying the coordinates and momenta at time zero is precisely that these initial conditions determine a classical solution – provided that the appropriate constraint equations are obeyed and modulo gauge transformations. Let us apply the principle that "phase space is the space of classical solutions"* to 2 + 1 dimensional gravity. The field equations (in the absence of a cosmological constant) assert the vanishing of the Ricci tensor, $R_{ij} = 0$, and in 2 + 1 dimensions this implies that space-time is flat. Thus, we are interested in flat space-times. To fix ideas, we further suppose that "space" is a Riemann surface Σ, of genus g; then "space-time" will be $\mathrm{M} = \Sigma \times \mathrm{R}^1$, where R^1 represents "time". We will look for flat metrics on M, but *not* necessarily for complete metrics. Part of the interest of the problem is precisely that the space-times that will arise upon solving the classical equations contain initial and final singularities; the implications, if any, of these singularities in the quantum theory are of much interest.

For illustrative purposes, I would like to give some concrete examples of flat space-times with initial singularities. Let Σ be a compact smooth two-dimensional surface of genus g, with no *a priori* complex structure assumed. It has a fundamental group $\Gamma = \pi_1(\Sigma)$. Let H be the complex upper half plane, with its natural metric of constant negative curvature. The group SL(2, R) acts on H, preserving this metric. Let Γ' be a subgroup of SL(2, R), isomorphic to Γ and acting discretely on H. Then H/Γ' is a Riemann surface of genus g, with a constant curvature metric (inherited from H) which gives it a complex structure.

Now, let X be 2 + 1 dimensional Minkowski space, with coordinates t, x, and y and metric $(\mathrm{d}s)^2 = -(\mathrm{d}t)^2 + (\mathrm{d}x)^2 + (\mathrm{d}y)^2$. Let X^+ be the interior of the future light cone, that is, the points of $t > 0$ and $t^2 - x^2 - y^2 > 0$. Let X^- be the interior of

* For the use of this principle to construct the canonical formalism in a manifestly covariant way, see refs. [8, 9].

the past light cone, consisting of points of $t<0$ and $t^2-x^2-y^2>0$. The $2+1$ dimensional Lorentz group is SO(2,1); the $2+1$ dimenisonal Poincaré group is ISO(2,1) (the "I" means that we are including the translations). A happy fact is that SO(2,1) and SL(2,R) are equivalent. Moreover, the hypersurface H′ in X^+, defined by

$$t^2-x^2-y^2=1, \tag{1.1}$$

(and $t>0$) is isomorphic with H. Thus, we can regard the group Γ' as a subgroup of SO(2,1) acting on H′; the quotient H'/Γ' is a Riemann surface of genus g. Now, to get a flat space-time, we consider Γ' to act not just on H′ but on the whole future light cone X^+. The quotient $Y^+=X^+/\Gamma'$ is flat, since X^+ is flat and Γ' preserves the metric of X. If we regard the hypersurfaces

$$t^2-x^2-y^2=\tau^2, \tag{1.2}$$

as surfaces of "equal time", with τ playing the role of "time", then the equal time slices of this flat space-time are Riemann surfaces of genus g. Eq. (1.2) describes an expanding universe, expanding from an initial singularity at $\tau=0$. Likewise, simply by considering X^-/Γ' instead of X^+/Γ', we can obtain flat space-times with a final singularity. These space-time models depend on the $6g-6$ real moduli of a Riemann surface of genus g (which enter in the choice of Γ').

Are these all of the flat space-times in which "space" is of genus g? Certainly not. The problem can be analyzed as follows. Let M be a flat space-time and let $\hat{M}$ be its simply connected universal cover. Being flat and simply connected, $\hat{M}$ is automatically isometric to Minkowski space, X, or perhaps to a subspace thereof. Let γ be a noncontractible loop in M; such a loop is a map of a circle into M such that $\gamma(\sigma+2\pi)=\gamma(\sigma)$. If such a loop is lifted up to $\hat{M}\subset X$, it does not close; it will only close modulo an isometry, that is, an element of ISO(2,1). Let us denote the element of ISO(2,1) associated in this way to a loop γ in M as $\phi(\gamma)$. It is easy to see that the map $\gamma\to\phi(\gamma)$ must be a homomorphism. Thus, flat structures on a given manifold M give homomorphisms of $\pi_1(M)$ into ISO(2,1). Conversely, given a homomorphism of $\pi_1(M)$ into ISO(2,1), the image of $\pi_1(M)$ is a subgroup Γ of ISO(2,1), and from this data we can reconstruct a flat three manifold, namely X/Γ, X being, as before, three-dimensional Minkowski space.

In our case, with $M=\Sigma\times R^1$, since R^1 is contractible, $\pi_1(M)$ reduces to $\pi_1(\Sigma)$. Thus, flat structures on $\Sigma\times R^1$ correspond more or less to homomorphisms of $\pi_1(\Sigma)$ into ISO(2,1). I say "more or less" because given a homomorphism, the space-time that one would reconstruct from it may have very nasty singularities. We have already given examples with initial and final singularities. Because of a rather non-trivial theorem that will be discussed in subsect. 3.1, certain even worse ailments, such as totally collapsed handles on the Riemann surface Σ, will not arise. (An interesting example of an exotic type of singularity which one might expect to run into when the vierbein and spin connection are independent variables is that discussed in ref. [10]. We will see later that this is one type of singularity that we will definitely have to allow.) Much of the interest in trying to quantize $2+1$ dimensional gravity is precisely the question of what class of objects should be considered in defining the "space of all classical solutions". In sect. 2, we will follow a simple canonical analysis which will lead us to consider the moduli space of *all* homomorphisms of $\pi_1(\Sigma)$ into ISO(2, 1).

How many parameters are required to specify a homomorphism of $\pi_1(\Sigma)$ (with Σ a Riemann surface of genus >1) into ISO(2, 1), or more generally, into any Lie group G? This question may easily be answered as follows. The fundamental group $\pi_1(\Sigma)$ is naturally described with $2g$ generators (the a and b cycles) and one relation. A homomorphism $\pi_1(\Sigma) \to \mathrm{G}$ can be described by giving $2g$ elements of G, one for each generator, obeying one relation. In addition, we must identify two homomorphisms if they differ by conjugation by an element of the group. This enables us, as far as counting parameters is concerned, to remove another element of G from the description of the homomorphism. Thus, the dimension of the moduli space is $2g-2$ times the dimension of G.

Various choices of G are of interest. The moduli space of homomorphisms of $\pi_1(\Sigma)$ to G, for $\mathrm{G}=\mathrm{SL}(2,\mathrm{R})$, is closely related to the moduli space of complex structures that can be put on Σ*. For $\mathrm{G}=\mathrm{ISO}(2,1)$ it is closely related to the space of flat structures on $\Sigma \times \mathrm{R}^1$. These are the two examples that we have discussed above. If we include a cosmological constant in general relativity, then Minkowski space is replaced by de Sitter space or anti-de Sitter space, and ISO(2, 1) is replaced by SO(3, 1) or SO(2, 2). The homomorphisms of $\pi_1(\Sigma)$ into one of these groups correspond more or less to the solutions of Einstein's equations with a cosmological constant of appropriate sign. If we replace the usual Einstein–Hilbert action of three-dimensional gravity with a pure Chern–Simons action, which is conformally invariant, then the symmetry ISO(2, 1) of Minkowski space is enlarged to the conformal group SO(3, 2). Homomorphisms of $\pi_1(\Sigma)$ into this group correspond more or less to conformally flat structures on $\Sigma \times \mathrm{R}^1$.

In the case of ISO(2, 1), since this group is six dimensional, the space of flat structures on $\Sigma \times \mathrm{R}^1$ has dimension $12g-12$, exactly double the dimension of the family that we found in the discussion surrounding eq. (1.2). The discrepancy obviously came from considering only the Lorentz transformations and not the translation generators in ISO(2, 1).

Even if there were no singularities to raise perplexing questions of principle about what we mean by "the space of classical solutions", it would not be satisfying to construct this space, formulate quantum mechanics on it, and dogmatically declare

* This is so because if we regard Σ as H/Γ', then a noncontractible loop on Σ lifts on Γ' to a loop that closes only modulo an SL(2, R) transformation. The moduli space of homomorphisms to SL(2, R) will be further discussed in subsect. 3.1

this to be the solution of $2+1$ dimensional gravity. One would like a field-theoretic analysis that naturally leads to the alleged quantum theory. Without basing our understanding of $\Sigma \times R^1$ on such a field-theoretic analysis, any understanding we might achieve of it would be purely isolated, unrelated to the study of other quantum field theories or to the study of the same theory on more elaborate three manifolds. So the bulk of this paper is devoted to a systematic field-theoretic analysis of quantum gravity on $\Sigma \times R^1$, the key insight being that the constraints of the canonical formalism* can be neatly untangled by making an equivalence of $2+1$ dimensional gravity with a suitable gauge theory. This gauge theory is like an ordinary Yang–Mills theory except that instead of the ordinary Yang–Mills interaction one has purely the Chern–Simons interaction. The vierbein formalism of general relativity makes general relativity temptingly similar to a gauge theory, and over the years many physicists have attempted to exploit this in different ways, e.g. refs. [14, 15]. Nevertheless, in four dimensions, gravity and gauge theory are definitely not equivalent. The surprise in the present paper is that we will find a precise equivalence of gravity and gauge theory in three dimeions. Our results can probably be regarded as a three-dimensional analog of recent proposals by Ashtekar for $3+1$ dimensional gravity [16]; in this work, an attempt has been made to give a gauge theory interpretation to the hamiltonian constraint equations of $3+1$ dimensional gravity. In related work, it has been proposed recently that $3+1$ dimensional gravity is related to knot theory [17]. What the future of this proposal will be in $3+1$ dimensions remains to be seen. But if the relation that we will allege between gravity and Chern–Simons gauge theory is valid at the quantum level, then there is a close relationship between gravity and knot theory at least in $2+1$ dimensions, since Chern–Simons gauge theory in that dimension is intimately connected with knot theory [18].

In sect. 2, we will discuss the canonical formalism of $2+1$ dimensional gravity at the classical level. In sect. 3, we consider quantization. Finally, in sect. 4, we discuss some additional topics, including some aspects of the motivation for this work that we have not touched on yet.

The canonical formalism of $1+1$ dimensional and $2+1$ dimensional gravity have been previously discussed in ref. [5]. In addition, after writing this paper, I learned of some earlier papers [6, 8] in which $2+1$ dimensional gravity is discussed somewhat along the lines of the above comments. We will try to extend their discussion in several ways, most crucially by showing how to actually solve the hamiltonian constraints of $2+1$ dimensional gravity and put the subject in a standard field-theoretic framework. This framework could be used, in principle, on an arbitrary three manifold though we will consider only three manifolds of the form $\Sigma \times R^1$ in this paper. Also, our viewpoint leads, as we will see at the end of sect. 3, to a straightforward renormalizable perturbation expansion for $2+1$ dimen-

* For the foundations of the canonical formalism of general relativity, see refs. [11–13].

sional quantum gravity. I also learned after writing this paper of new geometrical results [19] about the locally homogeneous lorentzian space-times that we will be studying.

2. Relation to the Chern–Simons action

We begin with general relativity on a space-time manifold M, of dimension d, which is to have lorentzian signature. We will denote tangent-space indices as i, j, k and "Lorentz" indices as a, b, c. It is convenient to describe general relativity in terms of a vierbein $e_i^{\ a}$ and a spin connection $\omega_i{}^a{}_b$. Geometrically, these have the following interpretation.

The smooth manifold M comes naturally with its tangent bundle T. We also introduce an abstract d-dimensional vector bundle V, with structure group $SO(d-1,1)$. (Saying that V has structure group $SO(d-1,1)$ is the same as saying that it is endowed with a metric, which we write as η_{ab}, of signature $(-++\dots+)$, and a volume form which we write $\epsilon_{a_1\dots a_d}$.) We suppose that V has the same topological type as T so that isomorphisms between V and T exist. However, there is no natural choice of an isomorphism. A vierbein e is a choice of isomorphism between T and V. It may also be regarded as a V-valued one form, obeying a certain condition of invertibility. The spin connection ω can be regarded as an $SO(d-1,1)$ valued connection on V. The isomorphism e and the connection ω can be regarded as the dynamical variables of general relativity.

The curvature tensor is defined as

$$R_{ij}{}^a{}_b = \partial_i\omega_j{}^a{}_b - \partial_j\omega_i{}^a{}_b + \left[\omega_i, \omega_j\right]^a{}_b\,, \tag{2.1}$$

or simply as $R = \mathrm{d}\omega + \omega\wedge\omega$. It can be regarded as a two form with values in $\wedge^2 V$. ($\wedge^k V$ will denote the kth antisymmetric tensor power (exterior power) of V.)

Let us consider, or instance, the case of $d=4$, which is the physical case at least macroscopically. The Einstein–Hilbert action can be written

$$I = \tfrac{1}{2}\int_M \epsilon^{ijkl}\epsilon_{abcd}\left(e_i^{\ a} e_j^{\ b} R_{kl}{}^{cd}\right). \tag{2.2}$$

This formula may be interpreted as follows. The expression $e\wedge e\wedge R$ is a four form on M with values in $V\otimes V\otimes \wedge^2 V$, which maps to $\wedge^4 V$. But since V, with structure group $SO(3,1)$, has a natural volume form, a section of $\wedge^4 V$ may be canonically regarded as a function. Thus, there is an invariantly defined integral $\int e\wedge e\wedge R$, and this is what is written in eq. (2.2).

To verify that eq. (2.2) is indeed the appropriate action for the Einstein theory of gravity, one proceeds as follows. The metric η_{ab} on V, together with the isomorphism $e_i^{\ a}$ between T and V, give a metric $g_{ij} = e_i^{\ a} e_j^{\ b}\eta_{ab}$ on T; this is the same as an ordinary metric on the manifold M. The connection ω, having structure group $SO(d-1,1)$, is metric compatible. Varying eq. (2.2) with respect to the connection ω one learns that

$$D_i e_j^a - D_j e_i^a = 0, \tag{2.3}$$

where D_i is the covariant derivative with respect to the connection ω. Eq. (2.3) precisely says that the metric compatible connection ω is also torsion free. These conditions uniquely identify ω as the riemannian or Levi–Civita connection associated with the metric g_{ij} on M. Finally, varying eq. (2.2) with respect to e we learn that

$$e^k{}_a = R_{ik}{}^a{}_b = 0 \tag{2.4}$$

(here $e^k{}_a$ is the inverse matrix of e). This is equivalent to vanishing of the Ricci tensor $R_{ij} = e_j{}^b e^k{}_a R_{ik}{}^a{}_b$. So eq. (2.4) is tantamount to the Einstein equations in vacuum.

Actually, there is a key limitation in the above argument. We assumed that the vierbein $e_i{}^a$ is invertible, so that the inverse matrix exists. This is related to the fact that in general relativity, the metric tensor $g_{ij} = e_i{}^a e_j{}^b \eta_{ab}$ is supposed to be non-degenerate. In fact, since the definition of the Riemann curvature tensor uses the inverse of g_{ij}, a configuration in which $e_i{}^a$ is not everywhere invertible must be regarded as a singularity in classical general relativity. This is precisely the type of singularity studied in ref. [10] (for the same reason – singularities of this type are very natural when the vierbein and connection are regarded as independent variables). Permitting $e_i{}^a$ to not be invertible may seem like a minor change, since the invertible $e_i{}^a$'s are in any case dense in the space of all possible $e_i{}^a$. However, if one attempted to project eq. (2.2) onto a subspace of invertible $e_i{}^a$'s, it would ruin the following discussion at crucial stages. We will see at the end of sect. 3 that in a sense the attempt to make such a projection is what leads to the alleged unrenormalizability of 2 + 1 dimensional gravity. So the only definite statement we will make in this paper about the role of singularities in quantum gravity is that from the point of view that we will develop, the type of "singularity" related to a non-invertible vierbein must be permitted to make sense of the quantum theory, at least in 2 + 1 dimensions.

2.1. THE VIERBEIN AS A GAUGE FIELD

In the last twenty years, many physicists have wished to combine together the vierbein $e_i{}^a$ and the spin connection $\omega_i{}^a{}_b$ into a gauge field of the group $ISO(d-1,1)$. The idea is that the spin connection would be the gauge field for Lorentz transformations, and the vierbein would be the gauge field for translations. One then tries to claim that "general relativity is a gauge theory of $ISO(d-1,1)$". However, there has always been something contrived about attempts to interpret general relativity as a gauge theory in that narrow sense. One aspect to the problem is that in four dimenisons, for instance, the Einstein action (2.2) is of the general form $\int e \wedge e \wedge (\mathrm{d}\omega + \omega^2)$. If we interpret e and ω as gauge fields, we should compare this to a gauge action $\int A \wedge A \wedge (\mathrm{d}A + A^2)$. But there is no such action in gauge theory. So we cannot hope that four-dimensional gravity would be a gauge theory in that sense.

In three dimenisons, the situation is rather different. For a space-time manifold M of dimension three, the Einstein–Hilbert action would be

$$I = \tfrac{1}{2}\int_{\mathrm{M}} \epsilon^{ijk}\epsilon_{abc}\left(e_i^{\ a}\left(\partial_j\omega_k^{\ bc} - \partial_k\omega_j^{\ bc} + [\omega_j, \omega_k]^{bc}\right)\right). \tag{2.5}$$

If we interpret the e's and ω's as gauge fields, this is of the general form $A\,\mathrm{d}A + A^3$, and might conceivably be interpreted as a Chern–Simons three form. The study of such terms in three-dimensional gauge theory has a relatively long history. Indeed, the Chern–Simons action in non-abelian 2 + 1 dimensional gauge theory was studied in refs. [20, 21], where it was considered as an additional term added to the unusual Yang–Mills action. In ref. [22], a quantization law associated with the Chern–Simons term was discovered. Abelian gauge theory with only the Chern–Simons term was studied by Schwarz [23] and in unpublished work by Singer; those authors related this theory to certain topological invariants (Ray–Singer analytic torsion). Non-abelian gauge theory with only the Chern–Simons interaction has recently turned out to be exactly soluble [18]. It is also interesting to note that string field theory can be formulated as a more abstract version of a 2 + 1 dimensional gauge theory with only a Chern–Simons action [24].

We will claim that three-dimensional general relativity, without a cosmological constant, is equivalent to a gauge theory with gauge group ISO(2, 1) and a pure Chern–Simons action.

Let us recall some facts about the Chern–Simons interaction. For a compact gauge group G, this may be written

$$I_{\mathrm{CS}} = \tfrac{1}{2}\int_{\mathrm{M}} \mathrm{Tr}\left(A \wedge \mathrm{d}A + \tfrac{2}{3}A \wedge A \wedge A\right). \tag{2.6}$$

Here we are regarding the gauge field A as a Lie-algebra-valued one form, and "Tr" really represents a non-degenerate invariant bilinear form on the Lie algebra.

Thus, if we choose a basis of the lie algebra, and write $A = A^a T_a$, then the quadratic part of eq. (2.6) becomes

$$\mathrm{Tr}(T_a T_b)\cdot\int_{\mathrm{M}}(A^a \wedge \mathrm{d}A^b). \tag{2.7}$$

Here $d_{ab} = \mathrm{Tr}(T_a T_b)$ plays the role of a metric on the Lie algebra, and this should be non-degenerate so that eqs. (2.6) or (2.7) contains a kinetic energy for all components of the gauge field.

Thus, before we ask whether gravity in 2 + 1 dimenisons is equivalent to ISO(2, 1) gauge theory with a Chern–Simons interaction, we should ask whether such a Chern–Simons interaction exists, or in other words whether there exists an invariant and non-degenerate metric on the Lie algebra of ISO(2, 1).

Let us first consider the general case of ISO($d-1$, 1). The Lorentz generators are J^{ab}, and the translations are P^a, with $a, b = 1, \dots, d$. A Lorentz-invariant bilinear expression in the generators would have to be of the general form $W = xJ_{ab}J^{ab} + yP_aP^a$, with some constants x and y. However, in requiring that W should commute with the P^b, we learn that we must set $x = 0$. At that point we are clearly no longer constructing a *non-degenerate* bilinear form on the Lie algebra, so there would be no reasonable Chern–Simons three form for ISO($d-1$, 1) for general d.

The magic of $d = 3$ is that in this case we can set $W = \epsilon_{abc}P^aJ^{bc}$. This is easily seen to be ISO(2, 1) invariant as well as non-degenerate. Therefore, a reasonable Chern–Simons action for ISO(2, 1) will exist. It remains to construct it and compare it to 2 + 1 dimensional general relativity.

For $d = 3$ it is convenient to replace J^{ab} with $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$. The invariant quadratic form of interest is then

$$\langle J_a, P_b \rangle = \delta_{ab}, \qquad \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0. \tag{2.8}$$

The commutation relations of ISO(2, 1) then take the form

$$[J_a, J_b] = \epsilon_{abc}J^c, \qquad [J_a, P_b] = \epsilon_{abc}P^c,$$
$$[P_a, P_b] = 0. \tag{2.9}$$

(The fact that this is ISO(2, 1) and not ISO(3) is hidden in the fact that it is the Lorentz metric that is used in raising and lowering indices. This will not always be indicated explicitly.)

Let us use these formulas and construct gauge theory for the group ISO(2, 1). The gauge field is a Lie-algebra-valued one form

$$A_i = e_i^a P_a + \omega_i^a J_a. \tag{2.10}$$

An infinitesimal gauge parameter would be $u = \rho^a P_a + \tau^a J_a$, with ρ^a and τ^a being infinitesimal parameters. The variation of A_i under a gauge transformation should be

$$\delta A_i = -\mathrm{D}_i u, \tag{2.11}$$

where by definition

$$D_i u = \partial_i u + [A_i, u]. \tag{2.12}$$

Upon evaluating eq. (2.10), we arrive at the transformation laws

$$\begin{aligned} \delta e_i{}^a &= -\partial_i \rho^a - \epsilon^{abc} e_{ib} \tau_c - \epsilon^{abc} \omega_{ib} \rho_c, \\ \delta \omega_i{}^a &= -\partial_i \tau^a - \epsilon^{abc} \omega_{ib} \tau_c. \end{aligned} \tag{2.13}$$

Now we calculate the curvature tensor

$$\begin{aligned} F_{ij} = [D_i, D_j] &= P_a\left(\partial_i e_j{}^a - \partial_j e_i{}^a + \epsilon^{abc}(\omega_{ib} e_{jc} + e_{ib}\omega_{jc})\right) \\ &\quad + J_a\left(\partial_i \omega_j{}^a - \partial_j \omega_i{}^a + \epsilon^{abc}\omega_{ib}\omega_{jc}\right). \end{aligned} \tag{2.14}$$

If now we were studying ISO(2, 1) gauge theory on a manifold without boundary of dimension four, we would form a topological invariant of the form $\int F^a \wedge F^b d_{ab}$ where d_{ab} is an invariant quadratic form on the Lie algebra. Using the quadratic form (2.8), we get for a four manifold Y the invariant

$$\begin{aligned} &\tfrac{1}{2}\int_Y \epsilon^{ijkl}\left(\partial_i e_j{}^a - \partial_j e_i{}^a + \epsilon^{abc}(\omega_{ib} e_{jc} + e_{ib}\omega_{jc})\right) \\ &\quad \times\left(\partial_k \omega_{la} - \partial_l \omega_{ka} + \epsilon_{ade}\omega_k{}^d \omega_l{}^e\right). \end{aligned} \tag{2.15}$$

Denoting the integrand in eq. (2.15) as U, a straightforward computation shows that U is a total derivative, $U = dV$. Therefore, if the four manifold Y has for its boundary a three manifold M, eq. (2.15) reduces to an integral on M. This integral is by definition the Chern–Simons action, and one easily finds it to be

$$I_{CS} = \int_M \epsilon^{ijk}\left(e_{ia}\left(\partial_j \omega_k{}^a - \partial_k \omega_j{}^a + \epsilon_{abc}\omega_j{}^b \omega_k{}^c\right)\right). \tag{2.16}$$

By this construction, eq. (2.16) is automatically invariant under the gauge transformations (2.13). In any case, this is easy to verify.

Now, a look back to eq. (2.5) reveals that the ISO(2, 1) Chern–Simons action (2.16) precisely coincides with the 2 + 1 dimensional Einstein action. Thus, we have essentially succeeded in showing that 2 + 1 dimensional gravity may be interpreted as Chern–Simons gauge theory. However, there is still an important point to clear up. The transformation laws (2.13) do not coincide with the usual transformation laws of 2 + 1 dimenisonal gravity. There is no problem with the local Lorentz transformations whose generators have been called τ^a in eq. (2.13); the terms in eq. (2.13) proportional to τ^a are the standard formulas for local Lorentz transformations. The problem is with the generators ρ^a which hopefully should be related to diffeomorphisms. From eq. (2.13) we see that under a transformation generated by the ρ's, the transformation law is

$$\delta e_i{}^a = -\partial_i \rho^a - \epsilon^{abc}\omega_{ib}\rho_c, \qquad \delta\omega_i{}^a = 0. \tag{2.17}$$

At first sight, eq. (2.17) does not seem to have much in common with the standard formulas for transformation under diffeomorphisms, but we want to show that they are equivalent. Under a diffeomorphism generated by a vector field $-v^i$, the standard transformation law would be

$$\tilde{\delta} e_i{}^a = -v^k\left(\partial_k e_i{}^a - \partial_i e_k{}^a\right) - \partial_i\left(v^k e_k{}^a\right),$$
$$\tilde{\delta}\omega_i{}^a = -v^k\left(\partial_k \omega_i{}^a - \partial_i \omega_k{}^a\right) - \partial_i\left(v^k \omega_k{}^a\right). \tag{2.18}$$

If we let $\rho^a = v^k e_k{}^a$, then we find that the difference between eqs. (2.17) and (2.18) is

$$\tilde{\delta} e_i{}^a - \delta e_i{}^a = -v^k\left(\mathrm{D}_k e_i{}^a - \mathrm{D}_i e_k{}^a\right) + \epsilon^{abc} v^k \omega_{kb} e_{ic}. \tag{2.19}$$

On the right-hand side of eq. (2.19), one term is proportional to

$$\mathrm{D}_k e_i{}^a - \mathrm{D}_i e_k{}^a, \tag{2.20}$$

and this term vanishes by the equations of motion. The remaining term on the right-hand side of eq. (2.19) is a local Lorentz transformation with infinitesimal parameter

$$\tau^a = v^k \omega_k{}^a. \tag{2.21}$$

Likewise, if one considers the variation of ω under transformations generated by ρ^a, eq. (2.17) is equivalent to eq. (2.18) on shell. To be precise, the variation of ω under local Lorentz transformations vanishes in eq. (2.17), while in eq. (2.18) it does not vanish identically but is equal on shell to a local Lorentz transformation with parameter (2.21).

Therefore, since local Lorentz transformations are in any case among the local symmetries of the problem, it does not matter whether one uses eq. (2.17) or eq. (2.18) as long as one is planning to impose the equations of motion (2.20). This is the key to the simplicity of three-dimensional gravity. All of the difficulties, both practical and conceptual, in studying the canonical formalism are associated with thinking of diffeomorphisms as operations that move the initial-value surface in an ambient space-time. The equivalence, on shell, of eqs. (2.17) and (2.18) means that in three-dimensional gravity it is not necessary to think in terms of constraints that generate motions of the initial-value surface. This greatly simplifies the whole conceptual framework of three-dimensional gravity. Above all it means that, just as in ordinary quantum mechanics, where the formula $e^{-iH(t_1+t_2)} = e^{-iHt_2} \cdot e^{-iHt_1}$ is a key part of our intuition, the transition amplitudes in three-dimensional gravity can be factored in sums over intermediate states. A transition from an initial state A to a final state C can be represented in terms of a sum over all possible intermediate states B that might be witnessed on some intervening spacelike hypersurface – just as in Yang–Mills theory.

2.2. INCLUSION OF A COSMOLOGICAL CONSTANT

We would now like to generalize this discussion to include a cosmological constant in three-dimensional gravity. The generalized lagrangian is

$$I = \int_{\mathrm{M}} \epsilon^{ijk}\left(e_{ia}\left(\partial_j \omega_k{}^a - \partial_k \omega_j{}^a \right) + \epsilon_{abc} e_i{}^a \omega_j{}^b \omega_k{}^c + \tfrac{1}{3}\lambda \epsilon_{abc} e_i{}^a e_j{}^b e_k{}^c \right). \qquad (2.22)$$

The equations of motion now say not that space-time is flat but that space-time is locally homogeneous, with curvature proportional to λ. The simply connected covering space of such a space-time is not a portion of Minkowski space, but a portion of de Sitter or anti-de Sitter space, depending on the sign of λ. These latter spaces have for their symmetries not ISO(2, 1) but SO(3, 1) and SO(2, 2), respectively. Thus, it is reasonable to guess that if three-dimensional gravity without a cosmological constant is related to gauge theory of ISO(2, 1), then three-dimensional gravity with a cosmological constant will be related to gauge theory of these latter groups. This proves to be the case.

To begin with, we generalize eq. (2.9) to

$$[J_a, J_b] = \epsilon_{abc} J^c, \qquad [J_a, P_b] = \epsilon_{abc} P_c, \qquad [P_a, P_b] = \lambda \epsilon_{abc} J^c. \qquad (2.23)$$

Introducing the gauge-field and covariant derivatives as in eqs. (2.10) and (2.12), we find that the transformation laws (2.13) generalize to

$$\begin{aligned} \delta e_i{}^a &= -\partial_i \rho^a - \epsilon^{abc} e_{ib} \tau_c - \epsilon^{abc} \omega_{ib} \rho_c, \\ \delta \omega_i{}^a &= -\partial_i \tau^a - \epsilon^{abc} \omega_{ib} \tau_c - \lambda \epsilon^{abc} e_{ib} \rho_c. \end{aligned} \qquad (2.24)$$

And formula (2.14) for the curvature is replaced by

$$\begin{aligned} F_{ij} = {} & P_a\left(\partial_i e_j{}^a - \partial_j e_i{}^a + \epsilon_{abc}\left(\omega_i{}^b e_j{}^c + e_i{}^b \omega_j{}^c \right) \right) \\ & + J_a\left(\partial_i \omega_j{}^a - \partial_j \omega_i{}^a + \epsilon^{abc}\left(\omega_{ib}\omega_{jc} + \lambda e_{ib} e_{jc} \right) \right). \end{aligned} \qquad (2.25)$$

The formula (2.8) gives an invariant quadratic form on the generalized Lie algebra (2.23). Using it, we find that the Chern–Simons three form comes out to be precisely the Einstein lagrangian (2.22) with cosmological constant included! The equations of motion derived from this lagrangian are precisely the vanishing of the field strength (2.25). Vanishing of the coefficient of P_a in eq. (2.25) is the assertion that ω is the Levi–Civita connection; and vanishing of the coefficient of J_a is then the Einstein equation with a cosmological constant.

2.3. A MORE GENERAL LAGRANGIAN

At this point it seems appropriate to point out a rather enigmatic feature of three-dimensional gravity. In addition to the invariant quadratic form (2.8), there is a second invariant quadratic form on the Lie algebra (2.23), namely

$$\langle J_a, J_b\rangle = \delta_{ab}, \qquad \langle J_a, P_b\rangle = 0, \qquad \langle P_a, P_b\rangle = \lambda\delta_{ab}. \tag{2.26}$$

Actually, eq. (2.26) is the specialization to $d=3$ of a quadratic form that would exist for any d. For $\lambda = 0$, eq. (2.26) is degenerate. This is the original reason that we considered eq. (2.8) instead. For $\lambda \neq 0$, the Lie algebra (2.23) is simple, and has only the unique invariant quadratic form (2.26), for every $d \neq 3$. The existence for $d=3$ of *two* invariant quadratic forms (2.8) and (2.26) is a consequence of the isomorphisms $SO(4) \simeq SU(2) \times SU(2)$, $SO(2,2) \simeq SL(2,R) \times SL(2,R)$. (Though $SO(3,1)$ does not undergo such a splitting over the real numbers, its complexification does split and this gives the two quadratic forms that we have noted.)

Beginning with the quadratic form (2.26) and following the usual steps, one arrives at the new Chern–Simons lagrangian

$$\tilde{I} = \int d^3x\, \epsilon^{jkl}\Big(\omega_j^a\big(\partial_k\omega_l^a - \partial_l\omega_k^a + \tfrac{2}{3}\epsilon_{abc}\omega_k{}^b\omega_l{}^c\big) + \lambda e_j{}^a\big(\partial_k e_l{}^a - \partial_l e_k{}^a\big) + 2\lambda\epsilon_{abc}\omega_j{}^a e_k{}^b e_l{}^c\Big). \tag{2.27}$$

Therefore, eq. (2.27) is invariant under eq. (2.24) and it makes sense to add it, with an arbitrary coefficient, to the original Einstein lagrangian (2.22). For generic values of this coefficient, the classical equations are unchanged – they still assert the vanishing of the field strength (2.25). This is rather strange – the more general lagrangian is equivalent to eq. (2.22) classically, but this will not be so quantum mechanically.

The situation can be clarified, perhaps, by introducing

$$J_a^{\pm} = \tfrac{1}{2}\left(J_a \pm \frac{1}{\sqrt{\lambda}} P_a \right). \tag{2.28}$$

Of course, this step only makes sense for $\lambda \neq 0$. If λ is negative, the J_a are complex. The Lie algebra (2.23) becomes simply

$$\begin{aligned} &[J_a^+, J_b^+] = \epsilon_{abc} J^{c+}, \qquad [J_a^-, J_b^-] = \epsilon_{abc} J^{c-}, \\ &[J_a^+, J_b^-] = 0. \end{aligned} \tag{2.29}$$

Obviously, for positive λ, eq. (2.29) is the Lie algebra of $SO(2,1) \times SO(2,1)$, or $SL(2,R) \times SL(2,R)$.

The corresponding connections are

$$A_i^{a\pm} = \omega_i^{\ a} \pm \sqrt{\lambda}\, e_i^{\ a}. \tag{2.30}$$

The covariant derivative (2.12) becomes simply

$$D_i = \partial_i + J_a^{\ +} A_i^{a+} + J_a^{\ -} A_i^{a-}. \tag{2.31}$$

The two Chern–Simons forms are simply

$$I^{\pm} = \int \epsilon^{ijk} \left(2A_i^{\ a\pm} \partial_j A_k^{\ a\pm} + \tfrac{2}{3} \epsilon_{abc} A_i^{a\pm} A_j^{b\pm} A_k^{c\pm} \right). \tag{2.32}$$

They are related as follows to the actions that we constructed earlier. The "standard" Einstein action (2.22) is $I = (I^+ - I^-)/4\sqrt{\lambda}$, and the "exotic" term (2.27) is $\tilde{I} = \frac{1}{2}(I^+ + I^-)$.

2.4. SUPERSYMMETRIC GENERALIZATION

We will now briefly indicate how these results extend to 2 + 1 dimensional supergravity. In supergravity, the groups ISO(2,1), SO(3,1), and SO(2,2) would be replaced by various supergroups. If the corresponding Lie superalgebras have invariant quadratic forms (symmetric in the graded sense), one could define Chern–Simons actions for these supergroups, and these Chern–Simons actions would serve as 2 + 1 dimensional supergravity actions. Thus, the issue comes down to whether the invariant quadratic forms that we have used on the Lie algebras of ISO(2,1), SO(3,1), and SO(2,2) have appropriate supergeneralizations.

To see that this is so, consider first the Lie algebra of SO(2,1), which is isomorphic to that of SL(2,R). This is the bosonic part of the superalgebra OSp(2|1) which is important in string theory. The latter has bosonic generators J_a and fermionic generators Q_α, transforming in the adjoint and spinor representations of

SO(2, 1), respectively. There is an invariant Casimir operator $J_a J^a + \epsilon^{\alpha\beta} Q_\alpha Q_\beta$, corresponding to the existence of an invariant (graded symmetric) bilinear form on the OSp(2|1) Lie algebra. (Here ϵ is the invariant antisymmetric tensor on the two-dimensional representation of SL(2, R).)

Now, considering three-dimensional gravity, if the cosmological constant is negative the bosonic group is SO(2, 2), which is the same as SO(2, 1) × SO(2, 1) or SL(2, R) × SL(2, R). Therefore, we can construct an "$N = 1$" supergravity theory based on SL(2, R) × OSp(2|1) or an "$N = 2$" theory based on OSp(2|1) × OSp(2|1). The appropriate Chern–Simons actions will exist in these two cases since from our above discussion the relevant quadratic forms on the Lie superalgebras do exist. By the Wigner–Inonu contraction, one could get 2 + 1 dimensional supergravity theories with zero cosmological constant and Chern–Simons actions. I do not know a similar construction with positive cosmological constant.

2.5. CHERN–SIMONS GRAVITY

We will conclude this section with a brief discussion of what would usually be called 2 + 1 dimensional gravity with a Chern–Simons action [22]. (The terminology is of course somewhat misleading since we are claiming that ordinary 2 + 1 dimensional gravity has a Chern–Simons interpretation.) Chern–Simons gravity means the following. The fundamental variable is a vierbein $e_i^{\ a}$. The spin connection is *defined* as a functional of $e_i^{\ a}$ by requiring it to obey

$$\mathrm{D}_i e_j^{\ a} - \mathrm{D}_j e_i^{\ a} = 0. \tag{2.33}$$

The lagrangian is then the Lorentz Chern–Simons three form

$$I' = \int_{\mathrm{M}} \epsilon^{ijk}\left(\omega_i^{\ a}\left(\partial_j \omega_k^{\ a} - \partial_k \omega_j^{\ a}\right) + \tfrac{2}{3}\epsilon_{abc}\omega_i^{\ a}\omega_j^{\ b}\omega_k^{\ c}\right). \tag{2.34}$$

I' is not varied with respect to ω regarded as an independent variable. Rather, one regards ω as a functional of e via eq. (2.33) and varies eq. (2.34) with respect to e. The field equation obtained in this way is

$$\mathrm{D}_i \bar{R}_{jk} - \mathrm{D}_j \bar{R}_{ik} = 0, \tag{2.35}$$

with $\bar{R}_{ij} = R_{ij} - \frac{1}{4} g_{ij} R$.

The lagrangian (2.34) – with ω defined in terms of e via eq. (2.33) – is invariant under local Weyl transformations $e_i^{\ a}(x, y, t) \to e^{\phi(x, y, t)} \cdot e_i^{\ a}$, even though this is not manifest in the way that eq. (2.34) is written. Consequently, eq. (2.35) is a conformally invariant equation. Indeed, the left-hand side of eq. (2.35) is the three-dimensional analogue of the Weyl tensor, and vanishing of eq. (2.35) is the condition asserting that space-time is conformally flat. Thus, the three-dimensional conformal group SO(3, 2) plays in eq. (2.34) the role that ISO(2, 1) plays in 2 + 1 dimensional general relativity without a cosmological constant. Therefore, it is natural to expect that eq. (2.34) is equivalent to an SO(3, 2) gauge theory with Chern–Simons action. To demonstrate this it is necessary to replace eq. (2.34) with an equivalent version in which eq. (2.33) would be an equation of motion rather than an independently imposed constraint. I will not attempt this here.

3. Quantization

We now turn to constructing a canonical formalism, with a view toward quantization. Thus, we consider the lagrangian (2.22) on a three manifold $M = \Sigma \times R^1$, with Σ being a Riemann surface that plays the role of an "initial-value surface". Some subtleties arise in the canonical formulation because of the gauge invariance. A convenient reference on the general procedure is ref. [25]. The first step in constructing a canonical formalism is to introduce new variables, if necessary, to get a lagrangian that is linear in time derivatives. We can skip this step, since eq. (2.22) is already linear in time derivatives. If possible, one then separates out the variables into variables whose time derivatives are present in the lagrangian and variables whose time derivatives do not appear*. In our case, this is easily done. The variables whose time derivatives appear in eq. (2.22) are the "spatial" components of the vierbein and connection, namely $e_i^{\ a}$ and $\omega_i^{\ a}$, for $i = 1, 2$. The variables whose time derivatives are absent in eq. (2.22) are the "time" components $e_0^{\ a}$ and $\omega_0^{\ a}$. This convenient, global separation between variables whose time derivatives appear in the lagrangian and variables whose time derivatives do not appear, and the fact that the lagrangian is linear in the latter, make the construction of a canonical formalism relatively straightforward.

Eq. (2.22) may be rewritten as

$$I = -2\int \mathrm{d}t \int_\Sigma \epsilon^{ij} e_{ia} \frac{\mathrm{d}}{\mathrm{d}t}\omega_j^{\ a} + \int \mathrm{d}t \int_\Sigma \Big(e_0^{\ a}\cdot \epsilon^{ij}\big(\partial_i\omega_j^{\ a} - \partial_j\omega_i^{\ a} + \epsilon^{abc}\omega_{ib}\omega_{jc} + \lambda\epsilon^{abc}e_{ib}e_{jc}\big) + \omega_0^{\ a}\cdot\epsilon^{ij}\big(\partial_i e_j^{\ a} - \partial_j e_i^{\ a} + \epsilon^{abc}(\omega_{ib}e_{jc} + e_{ib}\omega_{jc})\big)\Big). \tag{3.1}$$

* In discussing a closely related problem in ref. [18], I have adopted the possibly more familiar language of "picking the gauge $A_0 = 0$". In the gravitational problem that we are considering here, a different and perhaps more careful and canonical language seems appropriate.

The Poisson brackets can be read off from the terms in eq. (3.1) that contain time derivatives. They are

$$\left\{ \omega_i^{\,a}(x), e_j^{\,b}(y) \right\} = \tfrac{1}{2}\epsilon_{ij}\eta^{ab}\delta^2(x-y),$$

$$\left\{ e_{ia}(x), e_{jb}(y) \right\} = \left\{ \omega_{ia}(x), \omega_{jb}(y) \right\} = 0. \tag{3.2}$$

In addition, we must impose the constraint equations. They are simply the equations $\delta I/\delta e_0^{\,a} = \delta I/\delta \omega_0^{\,a} = 0$, or

$$\epsilon^{ij}\left(\partial_i e_j^{\,a} - \partial_j e_i^{\,a} + \epsilon^{abc}\left(\omega_{ib} e_{jc} + e_{ib}\omega_{jc} \right) \right) = 0,$$

$$\epsilon^{ij}\left(\partial_i \omega_j^{\,a} - \partial_j \omega_i^{\,a} + \epsilon^{abc}\left(\omega_{ib}\omega_{jc} + \lambda e_{ib} e_{jc} \right) \right) = 0. \tag{3.3}$$

Now, let us take stock of what these equations mean. Let G be the group ISO(2, 1) if $\lambda = 0$, and its generalization SO(3, 1) or SO(2, 2) if λ is not zero. It is natural to regard, $e_i^{\,a}$ and $\omega_i^{\,a}$, for $i = 1, 2$, as a gauge field on the Riemann surface Σ. The space of all such gauge fields, which we will call $\mathscr{W}$, is a phase space on which we have defined Poisson brackets (3.2). This is not yet the physical phase space, since it is necessary to impose the constraint equations (3.3).

Those constraint equations have a very natural interpretation. The objects which appear on the left of eq. (3.3) are precisely the curvatures or gauge-covariant field strengths constructed from the gauge field e, ω. Vanishing of these constraints means that we are dealing with a gauge connection which locally is a pure gauge; the only gauge-invariant observables that do not vanish when the constraints are imposed are global observables, such as holonomies around possible non-contractible loops in Σ.

It may be appropriate at this point to pause for a slight digression about the role of constraints in classical mechanics. In quantum mechanics, the "constraints" are operators, and they are imposed by declaring that a "physical state" is a state annihilated by the constraints. In our present problem, however, it is highly advisable to impose the constraints classically, *before* quantizing*. In classical physics, the imposition of the constraints is a two-step process. One begins with a phase space on which some Poisson brackets are defined. In addition, one has some constraints. These are certain functions on phase space. Let H^I denote the constraints. The first step in imposing the constraints classically is to declare that "physical values" of the canonical variables q^i are values for which $H^I = 0$. This is only half of the story, however. The second step is to interpret the constraints as the

* The problem of imposing these particular classical constraints to reduce this particular phase space was one element in the work of Atiyah and Bott on equivariant Morse theory, two-dimensional gauge fields, and the moduli space of holomorphic vector bundles [26]. Of course, 2 + 1 dimensional gravity gives a new context for this problem.

generators of certain transformations of phase space, via

$$\delta q^i = \sum \epsilon^I \{ H^I, q^i \}, \tag{3.4}$$

with the ϵ^I being infinitesimal parameters. Under favorable conditions, which prevail in the problem of interest to us in this paper, the transformations (3.4) exponentiate to the action of a group $\mathscr{G}$. The second step in imposing the constraints is to declare that two sets of "physical values" of the canonical variables are considered equivalent if they differ by an element of the group $\mathscr{G}$ generated by the constraints. Provided that the constraints generate a group, which we will call the constraint group, *the physical phase space of a constrained system is the space of solutions of the constraint equations modulo the action of the constraint group*. The Poisson brackets on the physical phase space are just the original Poisson brackets, restricted to functions that are invariant under the constraint group.

In the case at hand, the phase space is easily determined. The canonical variables $e_i^{\ a}$ and $\omega_i^{\ a}$ fit together into a G gauge field on Σ. The constraint equations (3.3) assert that this gauge field is a "flat connection", that is, the field strength vanishes. As for the group of transformations generated by the constraints, these are just gauge transformations! One may easily check, using the Poisson brackets (3.2), that the constraint operators that appear on the left of eq. (3.3) are the generators of the very gauge transformations that we have discussed in eq. (2.24)

$$\begin{aligned} \delta e_i^{\ a} &= -\partial_i \rho^a + \epsilon^{abc} e_{ib} \tau_c - \epsilon^{abc} \omega_{ib} \rho_c , \\ \delta \omega_i^{\ a} &= -\partial_i \tau^a - \epsilon^{abc} \omega_{ib} \tau_c - \lambda \epsilon^{abc} e_{ib} \rho_c . \end{aligned} \tag{3.5}$$

This should come as no surprise; in gauge theories, $\delta \mathscr{L} / \delta A_0$ is always the generator of gauge transformations. Thus, to construct the classical phase space which should be quantized, one simply takes the space of solutions of the constraints – namely the space of flat connections – and divides by the group generated by the constraints – namely, the group of gauge transformations. Consequently, the phase space $\mathscr{M}$ of 2 + 1 dimensional gravity is the same as the moduli space of flat G connections modulo gauge transformations.

The Poisson brackets on $\mathscr{M}$ are just the original Poisson brackets on $\mathscr{W}$, restricted to gauge-invariant functions. In other words, a physical observable is a function on $\mathscr{M}$. Modulo the constraints, functions on $\mathscr{M}$ are the same as gauge-invariant functions on $\mathscr{W}$, and the Poisson brackets of gauge-invariant functions on $\mathscr{W}$ are computed using the Poisson brackets (3.2).

In the introduction we have explained heuristically why the physical phase space of 2 + 1 dimensional gravity is related to the moduli space of flat G bundles. Now we have derived the result from a more conventional field-theoretic analysis. It remains to quantize the system.

3.1. REAL POLARIZATION

We have obtained a finite-dimensional phase space $\mathcal{M}$ with well-defined Poisson brackets, and we wish to quantize it. In general, this is not a straightforward operation. Contrary to the impression that may sometimes be given in elementary quantum mechanics, there is no general way to quantize a classical system (even if there are no global anomalies). The problems have to do with the topology of phase space – and, in the case at hand, the phase space $\mathcal{M}$ is definitely quite subtle topologically.

In practice, quantization usually requires a separation of the phase space variables into "coordinates" and "momenta". By definition, the "coordinates" are a maximal set of commuting variables. The quantum Hilbert space is then a suitable space of functions of the "coordinates". There actually are two important cases, the cases of a "real polarization" or a "complex polarization". The first corresponds to the case in which the phase space $\mathcal{M}$ is the cotangent bundle of some manifold $\mathcal{N}$. In that case the quantum Hilbert space is the space of square integrable functions on $\mathcal{N}$. The second case, of a "complex polarization", is the subject of subsect. 3.2.

Let us look at the Poisson brackets (3.2). The e_i are canonical conjugates of the ω_j. Naively, it appears that we are free to view the e_i as coordinates and the ω_j as momenta, or vice-versa. In the first case, the quantum Hilbert space $\mathcal{H}$ would be the space of functionals of the e_i, and in the second case it would be the space of functionals of the ω_j. This treatment is naive because it ignores the constraints. The constraints say that a physical state must be gauge invariant. A look back to eq. (3.5) shows that it is impossible for a wave functional that only depends on the e_i to be gauge invariant, since the gauge-transformation law includes a term $\delta e \sim \omega$. Another way to say this is that there is no such thing as "the space of all e_i modulo gauge transformations", since the gauge variation of the e_i depends on the values of the ω_j. Therefore, it is extremely awkward to impose the constraints if the e_i are regarded as coordinates.

What happens if, on the other hand, the ω_j are regarded as coordinates? In this case, for $\lambda \neq 0$, the situation is no better than before. But for $\lambda = 0$, we see in eq. (3.5) that the gauge-transformation law of ω_i is $\delta\omega \sim \omega$. The transformation law of ω is independent of e.

If the cosmological constant is zero, and we regard the ω_i as coordinates, it is very straightforward to impose the constraint equations*. ω_i can be interpreted as an SO(2, 1) (or equivalently SL(2, R)) connection on the Riemann surface Σ. If $\mathcal{Y}$ is the space of all such connections, then if there were no constraints to impose and we view the ω_i as the coordinates, the quantum Hilbert space would be the space of functions on $\mathcal{Y}$. Actually, we must incorporate the constraints. This is easy to do.

* The idea that it would be best to view the connection data as the coordinates was envisaged by Ashtekar [16] in 3 + 1 dimensions.

The second equation in (3.3) says that the curvature of the connection ω should vanish, and the second equation in (3.5) instructs us to identify two connections that differ by an SL(2, R) gauge transformation. Taken together, these conditions say that we should introduce the moduli space $\mathcal{N}$ of flat SL(2, R) connections modulo gauge transformations. The quantum Hilbert space, incorporating the constraints, is not the space of square integrable functionals on $\mathcal{Y}$ but the space of square integrable functionals on $\mathcal{N}$.

Geometrically, the situation may be described as follows. For zero cosmological constant, the relevant gauge group is ISO(2, 1). The ISO(2, 1) group manifold is the total space of the cotangent bundle of the SO(2, 1) manifold. Correspondingly, the moduli space $\mathcal{M}$ of flat ISO(2, 1) connections is the total space of the cotangent bundle of the moduli space $\mathcal{N}$ of flat SO(2, 1) connections.

Also, the Poisson brackets (3.2) induce on $\mathcal{M}$ its natural symplectic structure as the cotangent bundle of $\mathcal{N}$. Therefore, quantum mechanics on $\mathcal{M}$ is very simple – the quantum Hilbert space is the space of L^2 functions on $\mathcal{N}$.

For clarity, let me make this candidate for the "physical Hilbert space" of quantum gravity with zero cosmological constant completely explicit. The fundamental group of a Riemann surface Σ of genus $g > 1$ can be defined via $2g$ generators, which we denote

$$a_i, b_j, \quad i, j = 1, \dots, g, \tag{3.6}$$

with one relation,

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1. \tag{3.7}$$

A point in $\mathcal{N}$ is a homomorphism of the group with generators (3.6) and relation (3.7) into SO(2, 1). Such a homomorphism is described by representing the a_i and b_j by elements of SO(2, 1) which we will denote as U_i and V_j. These must obey

$$U_1 V_1 U_1^{-1} V_1^{-1} \dots U_g V_g U_g^{-1} V_g^{-1} = 1, \tag{3.8}$$

along with a topological condition that will be described presently. Two representations are equivalent if they differ by a global gauge transformation

$$U_i \to E^{-1} U_i E, \qquad V_j \to E^{-1} V_j E, \tag{3.9}$$

for some fixed element E of SO(2, 1). The quantum Hilbert space $\mathcal{H}$ is the space of functions $\Psi(U_i, V_j)$, such that: (i) Ψ is defined on the hypersurface that is defined by eq. (3.8) together with a certain topological condition that we will describe next; (ii) Ψ is invariant under the transformation (3.9).

By the way, the Hilbert space structure on these functions is defined by the norm $|\Psi|^2 = \int_{\mathcal{N}} \Psi * \Psi$. Evidently, for this to be invariant, Ψ must be a half-density on $\mathcal{N}$ rather than a "function". It is generally true that quantization leads naturally to spaces of half-densities rather than spaces of "functions". This subtlety is usually slurred over in quantum mechanics texts, something which is possible because in most physical problems there is an evident measure on coordinate space which can be used to give a canonical isomorphism between the space of half-densities and the space of functions. Even in the problem at hand, there are a variety of more or less natural measures on $\mathcal{N}$ which could be used to identify functions with half-densities, but since none of these has compellingly appeared in the above construction, it is most natural to think of the quantum Hilbert space $\mathcal{H}$ as the space of half-densities on $\mathcal{N}$.

3.1.1. A topological discursion. We must now discuss a certain subtle but important topological point, which we have suppressed until this point. Actually, the moduli space $\mathcal{N}$ of flat SO(2,1) connections on a Riemann surface Σ is not connected, but contains several components. These arise as follows. A flat SO(2,1) connection is of course the same as a flat SL(2, R) connection. Since SL(2, R) naturally acts on a real two-dimensional vector space, a flat SL(2, R) connection defines a real two-plane bundle $\mathscr{E}$ over Σ. Such bundles are classified topologically by an integer, the Euler class. In general, the Euler class of a real two-plane bundle $\mathscr{E}$ on a Riemann surface may have any integer value, but for a *flat* two-plane bundle, there is an upper bound – if $\mathscr{E}$ admits a flat SL(2, R) connection, its Euler class can be no bigger in absolute value than $2g-2$, which is the Euler class of the tangent bundle of a Riemann surface Σ of genus g.

Now, general relativity is supposed to be a theory of the dynamics of geometry. As we indicated in sect. 1, the relation of homomorphisms ϕ: $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathrm{R})$ to geometry is as follows. Suppose that the genus g of Σ is greater than one. (For genus zero there are no non-trivial homomorphisms to discuss; for genus one the situation is more complicated than the simple situation that I will now summarize for $g \geqslant 2$, and I will not attempt to discuss this case.) Let H denote the complex upper half plane. If ϕ *embeds* $\pi_1(\Sigma)$ as a subgroup Γ of SL(2, R), and if moreover Γ is a *discrete* subgroup of SL(2, R), then H/Γ is a complex Riemann surface with a complex structure determined by the homomorphism ϕ. All complex structures arise in this way, and the moduli space $\mathcal{N}'$ of homomorphisms (3.8) that give discrete embeddings of the fundamental group of Σ in SL(2, R) can be identified (if $g \geqslant 2$) with Teichmuller space. Thus, the quantum wave function $\Psi(U_i, V_j)$ described earlier can – if restricted to homomorphisms that give discrete embeddings of the fundamental group – be regarded as a function on Teichmuller space, or in other words as a function on conformal geometries.

However, it is far from being true that all homomorphisms ϕ: $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathrm{R})$ give discrete embeddings. The opposite of an embedding would be the trivial homomorphism $U_i = V_j = 1$ in eq. (3.8). If we consider "bad" homomorphisms of

$\pi_1(\Sigma)$ to SL(2, R), then SL(2, R)/Γ will be ill-behaved – non-compact, or with totally collapsed handles and other horrible singularities. At first sight, it appears that the construction of the quantum theory that we have advocated involves accepting much nastier singularities in the classical phase space than one might have wished.

What saves the day (and, I think, eliminates what would have been the main criticism of our alleged solution of quantum gravity in $2+1$ dimenisons) is the following remarkable theorem about Riemann surfaces. The "good" homomorphisms that give discrete embeddings of the fundamental group are precisely those that correspond to flat bundles of Euler class $2g-2$. A simple illustration of the singularities associated with flat bundles of Euler class less than $2g-2$ is the following; if a homomorphism ϕ is related to a flat bundle of Euler class $2g-2-2r$, then H/Γ may be a Riemann surface of genus g with r totally collapsed handles. (The trivial homomorphism $U_i = V_j = 1$ corresponds to a flat bundle of Euler class zero.)

Thus, the moduli space $\mathcal{N}$ of all homomorphisms $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathrm{R})$ has connected components corresponding to Euler class $2g-2, 2g-3, \ldots, -(2g-2)$. So the description of the quantum Hilbert space in subsect. 3.1 was incomplete. Bearing in mind that the negative values of the Euler class differ from the positive values by a reversal of orientation, there are essentially $2g-1$ possible Hilbert spaces, related to flat bundles of Euler class $0, 1, \ldots, 2g-2$. But there is exactly one choice that gives the sort of geometrical interpretation that we expect. This is the case of maximal Euler class $2g-2$, since the homomorphisms that give discrete embeddings are exactly those related to flat bundles with this value of the Euler class. Thus, with $\mathcal{N}'$ being the moduli space of homomoprhisms with the maximal Euler class, the quantum wave function $\Psi(U_i, V_j)$ of the last section should be regarded as a function on $\mathcal{N}'$. It is remarkable that the simple topological restriction on the Euler class eliminates the pathologies that seem to be present in the description of classical phase space that comes by untangling the constraint equations.

3.1.2. Closed timelike curves. Another issue that we can briefly discuss at this point is the question of closed timelike curves. The classical phase space that we are quantizing – the moduli space $\mathcal{M}$ of flat ISO(2, 1) bundles – definitely includes flat space-times with closed timelike curves. To quantize the system we have separated the coordinates and momenta in such a way that the "coordinates" label the space $\mathcal{N}'$ of flat SO(2, 1) bundles. Suddenly, the closed timelike curves are not much in evidence, since any point in $\mathcal{N}'$ corresponds to a sensible, "spacelike" Riemann surface. A quantum wave function $\Psi(U_i, V_j)$ does not correspond to any definite classical three geometry so that one cannot say that there definitely are or definitely are not closed timelike curves. Just as in any other quantum mechanics problem, one can construct quantum wave functions which, at least in some region of the parameter space, behave as if they are localized near any desired classical trajectory (in this case, any desired classical three geometry). It would be interesting to study this more carefully and see if there really are wave functions that correspond, with very high probability, to space-times with closed timelike curves.

3.1.3. Non-zero cosmological constant. We would now like to generalize this rather explicit description of the Hilbert space for $\lambda = 0$ to the case in which the cosmological constant is not zero. If the cosmological constant is positive, the relevant classical phase space $\mathcal{M}$ is the moduli space of homomorphisms of $\pi_1(\Sigma)$ to SO(3, 1). It can be shown [27] that this space is the total space of the cotangent bundle of the moduli space of homomorphisms of $\pi_1(\Sigma)$ to SO(3). Therefore, a real polarization is available, and the quantization can be carried out somewhat along the likeness of the above. However, the arguments are less elementary and will not be described here. As far as I know, it is not possible to find a real polarization when the cosmological constant is negative. There is another point of view about quantization of $\mathcal{M}$, which works for any value of the cosmological constant. This is what we will discuss next.

3.2. KAHLER POLARIZATION

The alternative approach to quantization is to put a Kahler structure on $\mathcal{M}$ and quantize it as a Kahler manifold. In our discussion so far, the "spatial" manifold Σ has simply been a smooth compact two-dimensional surface, with no other *a priori* structure. Let us now, purely as an aid in quantization, pick a complex structure J on Σ. When this is done, the moduli space $\mathcal{M}$ of flat G bundles becomes a Kahler manifold in a natural way. Once one picks a complex structure on Σ, the space $\mathcal{M}$ can be regarded as the moduli space of holomorphic vector bundles on Σ. Moreover, the symplectic form that can be inferred from the Poisson brackets (3.2) can be interpreted as the curvature form that represents the first Chern class of a certain holomorphic line bundle L over $\mathcal{M}$. According to standard principles of quantization, the quantum Hilbert space $\mathcal{H}$ can be identified as the space of holomorphic sections of the line bundle L.

This description of the quantum Hilbert space is not so explicit as the description that we gave, at the end subsect. 3.1, for the case of $\lambda = 0$. So one natural problem, which we will not tackle here, is to describe the quantum Hilbert space more explicitly. Another natural problem is to elucidate the relationship between the two descriptions of the Hilbert space in the case of zero or positive cosmological constant, where both descriptions are available.

3.3. 2 + 1 DIMENSIONAL GRAVITY AS A RENORMALIZABLE THEORY

Upon seeing that 2 + 1 dimensional gravity can be sensibly quantized, one is led to believe that there must be something wrong with the frequent assertion that it is an unrenormalizable theory. In fact, we will now address this point and claim that 2 + 1 dimensional gravity has a straightforward renormalizable perturbation expansion. Indeed, it is a theory with zero beta function.

Let us recall the lagrangian of 2 + 1 dimensional gravity

$$I = \frac{1}{\hbar}\int_{\mathrm{M}} \epsilon^{ijk}\left(e_{ia}\left(\partial_j\omega_k{}^a - \partial_k\omega_j{}^a\right) + \epsilon_{abc}e_i{}^a\omega_j{}^b\omega_k{}^c + \tfrac{1}{3}\lambda\epsilon_{abc}e_i{}^a e_j{}^b e_k{}^c\right). \tag{3.10}$$

We have restored Planck's constant $\hbar$ in eq. (3.10). By rescaling e (and redefining $\hbar$) one may assume that λ is 1, 0, or -1. In the units considered here (Newton's constant is one and λ is 1, 0, or -1), $\hbar$ is the only free parameter and is dimensionless. Perturbation theory – an expansion of the quantum theory in powers of the fields e and ω – will be valid for small $\hbar$.

One immediately sees that if one considers e and ω to be fields of dimension one, then the lagrangian (3.10) is renormalizable by power counting, since all terms in the lagrangian are of dimenison three. To actually construct perturbation theory, for quantization on a space-time manifold M, one picks a classical solution about which one wishes to expand. For example, one may expand about $e = \omega = 0$. In most physical theories, such a solution would be called the "trivial solution" corresponding to "unbroken symmetry". In general relativity, the solution with $e = 0$ might be considered unphysical. Our point of view is that this difference between general relativity and other theories is illusory and that just as in other theories the "trivial solution" with unbroken symmetry plays a key role. Just as in any other renormalizable gauge theory, the short-distance behavior is independent of the choice of classical solution to expand around, so one can not understand the expansion around any solution unless one can understand the "unbroken phase of general relativity", that is, the expansion around $e = \omega = 0$.

To actually construct this expansion, one needs gauge fixing. The gauge fixing may be carried out just as in Chern–Simons theories with compact gauge group [18, 23]. One picks an arbitrary "background" metric $g_{(0)}$ on M (unrelated to e or ω) and imposes the gauge condition

$$D_{(0)}{}^i e_i{}^a = D_{(0)}{}^i \omega_i{}^a{}_b = 0, \tag{3.11}$$

with $D_{(0)}{}^i$ denoting the covariant divergence with respect to $g_{(0)}$. The gauge condition (3.11) may be implemented by introducing Lagrange multipliers f_a, $\lambda_a{}^b$, and adding to eq. (3.10) the gauge fixing term

$$I' = \int_{\mathrm{M}} \sqrt{g_{(0)}}\left(f_a D_{(0)}{}^i e_i{}^a + \lambda_a{}^b D_{(0)}{}^i \omega_i{}^a{}_b\right). \tag{3.12}$$

In the usual fashion, this must be supplemented with Faddeev–Popov ghosts. The point of this gauge choice is that eq. (3.12) (and the ghost action) preserves the power-counting renormalizability of eq. (3.10).

Once it is accepted that eq. (3.10) leads to a renormalizable perturbative expansion, a variety of arguments show that the beta function must be zero. For one thing, the coupling constant in the Chern–Simons theory with compact gauge group is quantized, so it cannot be renormalized. The Feynman diagrams that come in quantizing eq. (3.10) are the same diagrams as the diagrams of the compact Chern–Simons theories, so they too must be finite. Another way to state much the same thing is that although eq. (3.10) is invariant under gauge transformations, it is not the integral of a gauge-invariant local functional. However, the structure of quantum perturbation theory shows that the counterterms, if any, are integrals of gauge-invariant local functionals, so the counterterms required to renormalize eq. (3.10) cannot include renormalization of the interactions that appear in eq. (3.10). Since there are in fact no other possible gauge interactions that are renormalizable by power counting, eq. (3.10) must lead to a finite quantum theory.

Given the simplicity of the above arguments, why is it usually felt that $2+1$ dimensional gravity is unrenormalizable? $2+1$ dimensional gravity appears unrenormalizable if the connection ω is eliminated and the theory is written entirely in terms of the metric $g_{ij} = e_{ia}e_j^{\ a}$. In terms of g_{ij}, the Einstein–Hilbert action is non-polynomial – in sharp contrast to the simple polynomial structure of eq. (3.10). Also, to write the Einstein–Hilbert action in terms of g_{ij}, one must introduce the inverse metric g^{ij}. In particular, when general relativity is written in terms of the metric (rather than the independent vierbein and spin connection), it is impossible to see the unbroken phase $g = 0$*. However, it is quite clear, in the renormalizable perturbation expansion sketched above, that as $e_i^{\ a}$ is a field of positive dimension, the short-distance behavior will involve the behavior in the unbroken phase $e = 0$. Thus, the fact that one has renormalizability in the description with e and ω is closely related to the fact that in this description one can see the unbroken phase $e = \omega = 0$. The usual attempt to quantize $2+1$ dimensional gravity in terms of g_{ij} is somewhat analogous to discussing a spontaneously broken gauge theory in a "unitary gauge" in which the underlying symmetry is not manifest. The attempt to quantize gauge theories in unitary gauge is notoriously treacherous.

It is amusing to think about $3+1$ dimensional gravity from this point of view. The lagrangian is of the general form

$$I_{(4)} \sim \int e \wedge e \wedge (\mathrm{d}\omega + \omega \wedge \omega) . \tag{3.13}$$

If one hopes for "power-counting renormalizability," one needs to assign dimension one to both e and ω, so that every term in eq. (3.13) is of dimension four. (Again, this is in contrast to the fact that the metric and vierbein are usually considered to

* This is related to the fact that usually, in discussions of the quantization of gravity, g is considered to be dimensionless, while in our treatment one must consider g to have dimension two, since e has dimension one.

have dimension zero.) As e and ω have positive dimension, the short-distance limit must have $e = \omega = 0$. The problem is now that as eq. (3.13) has no quadratic term in an expansion around $e = \omega = 0$, one cannot make sense of the "unbroken phase" that should govern the short-distance behavior; that is the essence of the unrenormalizability of quantum gravity in four dimensions.

4. Some additional considerations

The discussion of 2 + 1 dimensional gravity that we have given above raises many questions. In this concluding section I would like to briefly draw attention to a few of these questions, without claiming to solve any of them.

4.1. CONCEPTUAL PROBLEMS OF QUANTUM GRAVITY

One would certainly like to use 2 + 1 dimensional gravity as a probe to test some of the conceptual problems of quantum gravity. There are at least two outstanding problems that one would wish to investigate.

One concerns the singularities of quantum gravity. The classical phase spaces that we are quantizing include space-times with the initial and final singularities of the classical theory; it may be possible to say something interesting about the relation of these classical singularities to the quantum theory.

One is also interested in changes of topology in quantum gravity. It is not at all clear that the logical framework of quantum gravity requires one to consider processes in which the topology of space changes, but it is certainly interesting to try to calculate amplitudes associated with such processes. Until now, string theory, regarded as general relativity in 1 + 1 dimensions, is the only situation in which one knows how to do sensible calculations with change of topology in general relativity. However, in some other generally covariant theories, such as Gromov/Floer theory in 1 + 1 dimensions, Donaldson/Floer theory in 3 + 1 dimensions, and non-abelian Chern–Simons theory in 2 + 1 dimenisons, one has a sensible formalism in arbitrary space-time topology. The close relation of 2 + 1 dimensional general relativity to non-abelian Chern–Simons theory suggests that it will be possible to do calculations for processes with change of topology also in 2 + 1 dimensional relativity. Doing this will require extending the solution of the non-abelian Chern–Simons theory from the compact gauge groups that are related to the Jones polynomial to the non-compact groups, such as ISO(2, 1) and SO(3, 1), that are relevant to 2 + 1 dimensional general relativity.

4.2. EUCLIDEAN CONTINUATION

As we have just indicated, our formal arguments relating 2 + 1 dimensional gravity to Yang–Mills theory with a Chern–Simons term are not limited to $\Sigma \times R^1$. Formally, this connection should hold on an arbitrary three manifold. It seems natural to believe that it should hold even after any euclidean continuation that may be valid. But this raises puzzling issues.

Usually, the Einstein action is real whether one is in Minkowski or euclidean space. But the Yang–Mills Chern–Simons action is always imaginary in euclidean space. The question has to do with what is being continued when one goes from Minkowski to euclidean space. Usually, it seems obvious that in going from Minkowski to euclidean space the tangent space group of general relativity goes from SO(2, 1) to SO(3). However, in Yang–Mills theory one does not make a Wick rotation on the gauge group when one rotates from real to imaginary time.

Possibly, we should not think of $\Sigma \times R^1$, the manifold on which we have worked in this paper, as space-time, but rather as an auxiliary space analogous to the world sheet in string theory. The idea would be that space-time is reconstructed from data on $\Sigma \times R^1$ just as in string theory space-time is reconstructed from a world-sheet theory. The idea of this reconstruction is that a flat ISO(2, 1) connection on $\Sigma \times R^1$ is equivalent to a homomorphism of the fundamental group of Σ into ISO(2, 1). The image of the fundamental group under this homomorphism is a subgroup Γ of ISO(2, 1), and we try to identify spacetime with X/Γ, with X being Minkowski space. If we think of $\Sigma \times R^1$ as a "world sheet", and the dynamical variables e_i, ω_j as tools in reconstructing space-time, then as there is no metric on the world surface $\Sigma \times R^1$, there is no natural notion of whether this space has "Minkowski or euclidean signature". It *does* make sense to ask whether *space-time* has Minkowski or euclidean signature. The minkowskian case is the case, considered in this paper, in which e_i, ω_j are a gauge field of ISO(2, 1) or one of its generalizations SO(3, 1) or SO(2, 2). If we actually want to do euclidean gravity, meaning gravity with euclidean space-time, then those groups would be replaced with their analytic continuations ISO(3), SO(4), or SO(3, 1), depending on the sign of the cosmological constant.

A closely related question is whether the parameters in the gravitational lagrangian should be quantized, by analogy with the corresponding phenomenon in gauge theories [22]. This is inevitably related to the question of whether the lagrangian is to be real or imaginary, since it is only imaginary terms in the action that might be sensibly quantized. The standard Einstein action I of eq. (2.22) is ordinarily real in "euclidean space", and we would like to preserve this. Thus, it should not be quantized. The more exotic term I' of eq. (2.27) is less familiar and we are willing to believe that it should be quantized and perhaps should appear in the lagrangian with an imaginary coefficient. To investigate this, we note that quantization depends on $\pi_3(G)$, where G is ISO(2, 1) or one of its generalizations; and here there seems to be a big difference between Minkowski space-time and euclidean space-time, since for instance $\pi_3(\mathrm{ISO}(2,1))$ is zero, but $\pi_3(\mathrm{ISO}(3)) \sim \boldsymbol{Z}$. In fact, for all of the minkowskian groups and all of the euclidean groups except SO(4), there is no topological reason to quantize the ordinary Einstein action (2.22), and it can be given the usual real coefficient if we are considering the euclidean case. (SO(4) is from this point of view a mysterious exception that we will not try to elucidate.) However, for the minkowskian group SO(3, 1) and all of the euclidean groups, the exotic interaction I' is quantized and must appear in the lagrangian with an imaginary coefficient.

4.2.1. Global Anomalies and Classical Singularities. This whole discussion of global anomalies and quantization of couplings in $2+1$ dimensional gravity may seem rather bizarre. When $2+1$ dimensional gravity is written in terms of the metric (rather than vierbein and connection), it is manifest that there are no such anomalies. So what is going on? Actually, the crucial point is that (as discussed at the end of the introduction to sect. 2), in formulating $2+1$ dimensional gravity in terms of the vierbein and spin connection, we have dropped the requirement that the vierbein should be invertible. Though the non-invertible vierbeins are of "measure zero", adding them changes the topology of field space (and of the space of gauge transformations) and permits the occurrence of global anomalies that otherwise would have been absent. In fact, the four-dimensional "instanton" studied in ref. [10], which has a classical singularity (a degenerate vierbein) at its core, is precisely the configuration which is manifested in three space-time dimensions in terms of global anomalies.

4.3. UNITARITY

Some of the fundamental puzzles in the canonical formalism of quantum gravity have to do with the physical interpretation of the Wheeler–de Witt wave function. The following may illustrate some of the questions. Let P_{BA} be the probability amplitude for observing a final state B after having observed an initial state A (the initial and final observations being on some specified spacelike hypersurfaces). It is a fundamental fact of life in ordinary field theory that in a sequence of observations (fig. 1) one has

$$P_{\mathrm{CA}}=\sum_{\mathrm{B}} P_{\mathrm{CB}}\cdot P_{\mathrm{BA}}. \qquad (4.1)$$

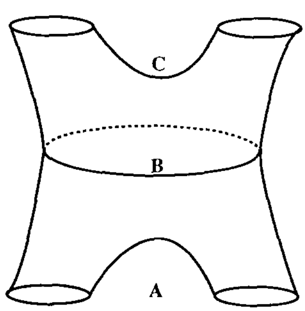

Fig. 1. A two step transition from an initial state A to a final state C via an intermediate state B.

In ordinary quantum mechanics this is essentially the statement that

$$e^{-iH(t_1+t_2)} = e^{-iHt_2} \cdot e^{-iHt_1}. \tag{4.2}$$

In quantum gravity it appears problematical to formulate such a relationship. On the right-hand side one would need an integral over the elapsed time from A to B, and another integral over the elapsed time from B to C. It is hard to see how this can agree with a single integral over elapsed time from A to C, required on the left.

Nevertheless, there are generally covariant field theories that behave as in eq. (4.1). An example is provided by Donaldson/Floer theory in 3 + 1 dimensions; for a discussion in physical language see ref. [28]. Another, perhaps even more surprising example is 2 + 1 dimensional gravity. For, being equivalent to an ordinary gauge theory, 2 + 1 dimensional gravity certainly obeys such relations as eq. (4.1). I have no idea whether (i) despite appearances, all generally covariant theories can be formulated in such a way as to ensure the validity of eq. (4.1); or (ii) the validity of such a formula should be viewed as an order parameter distinguishing an unbroken phase of general relativity (which would include Donaldson theory and 2 + 1 dimensional gravity) from the ordinary broken phase of general relativity that we experience in the real world.

4.4. OBSERVABLES

Another interesting question has to do with the observables in quantum gravity. The whole question of observables in quantum gravity is rather thorny. In 2 + 1 dimensional gravity, there is a peculiar class of observables that would not have a good counterpart in any other dimension. Let C be a circle in space-time. In three space-time dimensions, a circle in space-time may be "knotted".

Let us denote as A, the gauge field, with gauge group ISO(2, 1), SO(3, 1), or SO(2, 2), that is obtained, as in eq. (2.10), by combining together e and ω. Let $\mathcal{R}$ be a representation of one of those groups (a finite-dimensional representation or in general any representation such that the observable defined below makes sense in the quantum theory). For every circle C in space-time and every representation $\mathcal{R}$, define

$$O_{\mathcal{R}}(\mathrm{C}) = \mathrm{Tr}_{\mathcal{R}} P \exp \oint_{\mathrm{C}} A_i \, \mathrm{d}x^i. \tag{4.3}$$

The symbol $\mathrm{Tr}_{\mathcal{R}}$ represents a trace in the $\mathcal{R}$ representation, and $P\exp\oint A_i\,\mathrm{d}x^i$ denotes the holonomy of the connection A around the loop C.

Because of general covariance, expectation values

$$\left\langle \prod_i O_{\mathcal{R}_i}(C_i) \right\rangle \tag{4.4}$$

should formally depend on the topological classes of the links C_i. Any two unknotted circles on the three sphere are equivalent, for example. One might wonder whether such observables can actually be expected to make sense. *A priori* one might well have been inclined toward a negative answer, but some recent developments relating Yang–Mills theory to knot theory [18] strongly suggest a positive answer. If the equivalence that we have proposed between quantum gravity in $2+1$ dimenisons and Chern–Simons gauge theory with gauge group ISO(2, 1), SO(3, 1), or SO(2, 2) is really correct, then the expectation values (4.4) should be the analogues for these groups of the Jones polynomials [29] of knot theory. This would be a $2+1$ dimensional version of the possible relation between quantum gravity and knot theory conjectured in ref. [17].

4.5. IS 2 + 1 DIMENSIONAL GRAVITY RELATED TO THE VIRASORO DISCRETE SERIES?

Another fascinating cluster of questions concerns an important motivation for the present work which so far we have not even mentioned.

Some recent developments [18] suggest that important classes of conformally invariant quantum field theories in $1+1$ dimensions can be obtained from generally covariant quantum field theories in $2+1$ dimensions. The Virasoro discrete series with $c<1$ are a very distinguished class of $1+1$ dimensional conformally invariant theories. Therefore, it seems natural to suspect that some distinguished generally covariant theories in $2+1$ dimensions should underlie the Virasoro discrete series.

What $2+1$ dimensional generally covariant theory is more distinguished than gravity itself? So one is tempted to believe that the Virasoro discrete series can be understood by quantizing general relativity in $2+1$ dimensoins.

How close to this goal have we come? We have seen that $2+1$ dimensional gravity is closely related to the Chern–Simons gauge theory of certain groups. In view of the results of [18], this means that $2+1$ dimensional gravity is closely related to current algebra theories in $1+1$ dimensions with these symmetry groups. If we work in three-dimensional euclidean space and the cosmological constant is positive, then the group that arises is $SU(2)\times SU(2)$. It is certainly true, in view of the GKO coset space construction, that $SU(2)\times SU(2)$ current algebra is closely related to the Virasoro discrete series. But in this paper we have not uncovered a rationale for modifying the $SU(2)\times SU(2)$ theory with the coset space construction. If such a rationale does exist, it is probably related to the shakiest part of our construction, which is that we took the phase space to be the moduli space of *all* flat G connections without imposing any condition to avoid singularities.

4.5.1. String Theory Interpretation of 2 + 1 Dimensional Gravity. Whether or not $2+1$ dimensional gravity is related to the Virasoro discrete series, it certainly looks temptingly like it has a string theory interpretation. For, according to ref. [18], the Chern–Simons theories to which $2+1$ dimensional gravity are equivalent are in turn closely related to certain $1+1$ dimensional conformal field theories. This latter relationship comes essentially from interpreting the quantum Hilbert spaces in $2+1$ dimensions as the spaces of conformal blocks in $1+1$ dimensions; the quantum Hilbert spaces of the $1+1$ dimensional theory can also be obtained by quantizing the $2+1$ dimensional theory on a manifold with boundary. So we have come upon a peculiar relative of what one aims to do in string theory: a theory of space-time geometry ($2+1$ dimensional gravity, in this case) can be coded in a $1+1$ dimensional conformal theory.

4.5.2. Thickening The Moduli Space of Riemann Surfaces. On noting that $2+1$ dimensional gravity makes sense, and is "trivial", just like the $1+1$ dimensional theory, one might think of imitating the role of $1+1$ dimensional gravity in string theory. However, this is presumably naive. Just as unexpected as the role of $1+1$ dimensional gravity in string theory may appear, so the role in physics of $2+1$ dimensional gravity – if there is any role – is likely to be quite unexpected.

In speculating about possible relations between $2+1$ dimensional gravity and string theory, it is interesting to note that the moduli spaces that appear in $2+1$ dimensional gravity are closely related to moduli space of Riemann surfaces. It is for this reason that the study of hyperbolic three-dimensional geometry has been closely connected with the study of Riemann surfaces. (An introduction can be found in ref. [30].) It is intriguing to conjecture that suitable moduli spaces of hyperbolic three geometries will enter string theory in the future as thickened versions of an ordinary Riemann-surface moduli space. The behavior of a three geometry in the far past and the far future may give separate Riemann-surface moduli for left- and right-moving modes. There is some heuristic indication that such a separation is needed in string theory [31].

4.6. GEOMETRICAL APPLICATIONS OF QUANTUM GRAVITY

In this paper, we have considered $2+1$ dimensional gravity from a physical point of view. However, if it is really possible to generalize the solution of the Yang–Mills Chern–Simons theory to non-compact groups and understand quantum gravity on general three manifolds, this is likely to have important implications for geometry. The most important case for geometrical applications, in view of the results of geometers [30], is likely to be the case of euclidean signature with negative cosmological constant – the relevant gauge group is then SO(3, 1). In this case, the lagrangian is

$$\hat{I} = \frac{1}{\hbar} I + \frac{ik}{8\pi} I', \tag{4.5}$$

where I is the standard Einstein action (2.22) with cosmological constant, and I' is the exotic action (2.27). In keeping with the above discussion of quantization of couplings, the standard action I appears with a real coefficient, which we have written as $1/\hbar$; here $\hbar$ is Planck's constant. But – again in view of the discussion above $-I'$ has a quantized coefficient, with k an integer. Once $\hbar$ is explicitly introduced in this way, one may as well set $\lambda = 1$ in eq. (2.22). Now one wishes to study the Feynman integral over all choices of field variables on an arbitrary three manifold M, to get the "partition function" defined by

$$Z(\mathrm{M}) = \int \mathrm{D}e\,\mathrm{D}\omega\, \mathrm{e}^{-\hat{I}}. \tag{4.6}$$

Understanding quantum gravity on a general three manifold would mean understanding how to compute $Z(\mathrm{M})$ as a function of the variables $\hbar$ and k that appear in the lagrangian.

The connection with classical geometry should be particularly striking in the limit of small $\hbar$. Presumably, though this is part of what one would want to investigate, the small-$\hbar$ limit of the partition function would be dominated by the classical solution of most negative action. According to the standard conjectures about three manifolds, almost all interesting (irreducible) three manifolds are "hyperbolic", and the action (4.5) would have a unique non-trivial critical point of most negative action up to gauge transformation. The action for this critical point is $-(V/\hbar + 2\pi ikC)$, where V and C are known as the volume and Chern–Simons invariant of the hyperbolic three manifold. The small-$\hbar$ limit of the partition function would be $Z \sim \exp(V/\hbar + 2\pi ikC)$ (up to a power of $\hbar$), so that the classical invariants V and C could be extracted from the asymptotic behavior of Z, if indeed it is possible to define the partition function Z as an invariant of three manifolds. Notice that, since quantum gravity can be formulated on three manifolds that are not necessarily hyperbolic, the scenario just sketched would be impossible if the hyperbolic volume were well defined only for hyperbolic three manifolds. Happily, Gromov has shown that the definition of the hyperbolic volume can be extended to arbitrary three manifolds; and his definition seems rather like it is related to the classical limit of quantum gravity.

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