Black hole entropy from near-horizon microstates



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# Black hole entropy from near-horizon microstates

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Abstract: Black holes whose near-horizon geometries are locally, but not necessarily globally, $AdS_3$ (three-dimensional anti-de Sitter space) are considered. Using the fact that quantum gravity on $AdS_3$ is a conformal field theory, we microscopically compute the black hole entropy from the asymptotic growth of states. Precise numerical agreement with the Bekenstein-Hawking area formula for the entropy is found. The result pertains to any consistent quantum theory of gravity, and does not use string theory or supersymmetry.

Keywords: Black Holes in String Theory, Conformal and W Symmetry, Black Holes.

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## Contents





# 1. Introduction

The idea that the black hole entropy should be accounted for by microstates near the black hole horizon has great appeal and a long history [1, 2, 3, 4, 5, 6, 7, 8, 9]. One reason for this is that a demonstration that the horizon has of order one degree of freedom per Planck area would provide a statistical explanation of the area formula for the entropy. Such a picture might also shed light on the information puzzle, in that the information is more safely stored on the surface than in the causally inaccessible black hole interior.

While much has been learned, attempts at a precise statistical accounting for the entropy along these lines have been hampered both by the ultraviolet problems in quantum gravity as well as the infinite number of low-energy modes which are nevertheless high-frequency because of the large near-horizon redshifts[1]. In practice it has not been clear which modes should or should not be counted, and the results appear cutoff dependent.

Recently a statistical derivation of the black hole entropy has been given for some cases in string theory [10]. This derivation employed a continuation to a weak-coupling region where the black hole is treated as a pointlike object and its microstates are counted by a certain conformal field theory. This construction did not directly address

[1]With the notable exception of the topological theory of gravity considered in [7] and discussed below.

the issue of whether or not, in regimes for which the black hole is not effectively pointlike and has a clear horizon, the microstates are in any sense near the horizon.

In this paper we will address this issue in the context of black holes whose near-horizon geometry is locally $AdS_3$. This includes many of the string theory examples and the three-dimensional BTZ black hole [11]. We will microscopically derive the black hole entropy by counting excitations of $AdS_3$. Our derivation relies only on general properties of a diffeomorphism-invariant theory and will not use string theory or supersymmetry.

Our basic result follows quickly from prior results in the literature. Some time ago, Brown and Henneaux [12] made the remarkable observation that the asymptotic symmetry group of $AdS_3$ is generated by (two copies of) the Virasoro algebra, and that therefore *any consistent quantum theory of gravity on $AdS_3$ is a conformal field theory.*[2] They further computed the value of the central charge as $c = \frac{3}{2G\sqrt{-\Lambda}}$, where $G$ is Newton's constant and $\Lambda$ is the cosmological constant. In this paper we simply apply Cardy's formula [14] for the asymptotic growth of states for a conformal field theory of central charge $c$ to microscopically compute the black hole entropy. Precise agreement with the Bekenstein-Hawking area formula is found.

The conformal field theory that describes the black hole microstates lives on a $(1+1)$-dimensional cylinder surrounding the black hole. Since all the information is encoded in this cylinder and remains outside the horizon there is no information loss in this picture.

This paper is organized as follows: In Section 2 the result of Brown and Henneaux is reviewed. In Section 3 the BTZ black hole is reviewed. In Section 4 other relevant black holes which approach $AdS_3$ near the horizon are discussed. In Section 5 the entropy is microscopically computed. In Section 6 we relate the present work to other derivations of the black hole entropy as well as Maldacena's recent work on near-horizon dynamics in string theory [13]. We close with discussion in Section 7.

## 2. Quantum gravity on $AdS_3$ as a conformal field theory

In this section we review the results of [12]. Three-dimensional gravity coupled to matter is described by the action

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R + \frac{2}{\ell^2}) + S_m\,, \tag{2.1}$$

where $S_m$ is the matter action, the cosmological constant is $\Lambda = -\frac{1}{\ell^2}$ and we have omitted surface terms. The matter action will not play an important role in the following, but we include it here to stress the generality of our considerations. In the following we will be interested in a semiclassical description, which requires that the cosmological

[2]Maldacena has recently shown, from a totally different perspective, that certain string compactifications to $AdS_3$ are conformal field theories [13].



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constant is small in Planck units, or equivalently

$$\ell \gg G\,. \tag{2.2}$$

The action (2.1) has the AdS$_3$ vacuum solution

$$ds^2 = -\left(\frac{r^2}{\ell^2}+1\right)dt^2 + \left(\frac{r^2}{\ell^2}+1\right)^{-1}dr^2 + r^2 d\phi^2\,, \tag{2.3}$$

where $\phi$ has period $2\pi$. AdS$_3$ is the $SL(2,R)$ group manifold and accordingly has an $SL(2,R)_L \otimes SL(2,R)_R$ isometry group. In order to define the quantum theory on AdS$_3$, we must specify boundary conditions at infinity. These should be relaxed enough to allow finite mass excitations and the action of $SL(2,R)_L \otimes SL(2,R)_R$, but tight enough to allow a well-defined action of the diffeomorphism group. This requires [12] [3]

$$\begin{aligned}
g_{tt} &= \frac{-r^2}{\ell^2} + \mathcal{O}(1),\\
g_{t\phi} &= \mathcal{O}(1)\,,\\
g_{tr} &= \mathcal{O}(\frac{1}{r^3})\,,\\
g_{rr} &= \frac{\ell^2}{r^2} + \mathcal{O}(\frac{1}{r^4}),\\
g_{r\phi} &= \mathcal{O}(\frac{1}{r^3}),\\
g_{\phi\phi} &= r^2 + \mathcal{O}(1)\,.
\end{aligned} \tag{2.4}$$

Allowed diffeomorphisms are generated by vector fields $\zeta^a(r,t,\phi)$ which preserves (2.4). These are of the form

$$\begin{aligned}
\zeta^t &= \ell(T^+ + T^-) + \frac{\ell^3}{2r^2}(\partial_+^2 T^+ + \partial_-^2 T^-) + \mathcal{O}(\frac{1}{r^4})\,,\\
\zeta^\phi &= T^+ - T^- - \frac{\ell^2}{2r^2}(\partial_+^2 T^+ - \partial_-^2 T^-) + \mathcal{O}(\frac{1}{r^4})\,,\\
\zeta^r &= -r(\partial_+ T^+ + \partial_- T^-) + \mathcal{O}(\tfrac{1}{r})\,,
\end{aligned} \tag{2.5}$$

where $2\partial_\pm \equiv \ell\dfrac{\partial}{\partial t} \pm \dfrac{\partial}{\partial \phi}$ and preservation of (2.4) requires that $T^\pm$ depend on $r$, $\phi$ and $t$ only as $T^\pm(r,t,\phi) = T^\pm(\frac{t}{\ell} \pm \phi)$ so that $\partial_\pm T^\mp = 0$.

Diffeomorphisms with $T^\pm = 0$ fall off rapidly at infinity and should be considered "pure gauge transformations". In the quantum theory the corresponding generators will annihilate physical states. The diffeomorphisms with nonzero $T_\pm$ modulo the pure gauge transformations comprise the asymptotic symmetry group. Let us denote the generators of these diffeomorphisms by

$$L_n, \qquad \bar{L}_n \qquad -\infty < n < \infty\,,$$

[3]Matter fields, if present, are assumed to fall off rapidly enough so as not to affect the asymptotic form of the symmetry generators.

where $L_n$ ($\bar{L}_n$) generates the diffeomorphism with $T^+ = e^{in(\frac{t}{\ell}+\phi)}$ $\left(T^- = e^{in(\frac{t}{\ell}-\phi)}\right)$. The generators obey the algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}\,, \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}\,, \\
[L_m, \bar{L}_n] &= 0\,,
\end{aligned} \tag{2.6}$$

with

$$c = \frac{3\ell}{2G}\,. \tag{2.7}$$

The explicit computation of the central charge $c$ proceeds [12] by taking the matrix element of (2.6) in the $\mathrm{AdS}_3$ vacuum (2.3). The central charge is then related to an integral of the vector field parameterizing the diffeomorphism generated by $L_m$ over the vacuum (2.3) perturbed by $L_{-m}$. We note that in the semiclassical regime (2.2)

$$c \gg 1\,. \tag{2.8}$$

(2.6) is of course the Virasoro algebra. Since the physical states of quantum gravity on $\mathrm{AdS}_3$ must form a representation of this algebra, we have the remarkable result [12] *quantum gravity on $AdS_3$ is a conformal field theory with central charge* $c = \frac{3\ell}{2G}$. The conformal field theory lives on the $(t, \phi)$ cylinder at spatial infinity.

## 3. The BTZ black hole

An important example to which our considerations apply is the BTZ black hole [11, 15]. In this section we recall a few of its salient features. Further details can be found in [15].

The metric for a black hole of mass $M$ and angular momentum $J$ is

$$ds^2 = -N^2dt^2 + \rho^2(N^\phi dt + d\phi)^2 + \frac{r^2}{N^2\rho^2}dr^2\,, \tag{3.1}$$

with

$$\begin{aligned}
N^2 &= \frac{r^2(r^2 - r_+^2)}{\ell^2\rho^2}\,, \\
N^\phi &= -\frac{4GJ}{\rho^2}\,, \\
\rho^2 &= r^2 + 4GM\ell^2 - \frac{1}{2}r_+^2\,, \\
r_+^2 &= 8G\ell\sqrt{M^2\ell^2 - J^2}\,,
\end{aligned} \tag{3.2}$$

where $\phi$ has period $2\pi$. These metrics obey the boundary conditions (2.4), and the black holes are therefore in the Hilbert space of the conformal field theory. The Bekenstein-Hawking black hole entropy is

$$S = \frac{\text{Area}}{4G} = \frac{\pi\sqrt{16GM\ell^2 + 2r_+^2}}{4G}\,. \tag{3.3}$$

It is convenient to choose the additive constants in $L_0$ and $\bar{L}_0$ so that they vanish for the $M = J = 0$ black hole. One then has

$$M = \frac{1}{\ell}(L_0 + \bar{L}_0)\,, \tag{3.4}$$

while the angular momentum is

$$J = L_0 - \bar{L}_0\,. \tag{3.5}$$

The metric for the $M = J = 0$ black hole is

$$ds^2 = -\frac{r^2}{\ell^2}dt^2 + r^2 d\phi^2 + \frac{\ell^2}{r^2}dr^2\,. \tag{3.6}$$

This is *not* the same as $\mathrm{AdS}_3$ metric (2.3) which has negative mass $M = -\frac{1}{8G}$. Locally they are equivalent since there is locally only one constant curvature metric in three dimensions. However they are inequivalent globally. The family of metrics (3.1) can be obtained from (2.3) by a discrete identification followed by a coordinate transformation. Hence the black holes are topologically inequivalent to $\mathrm{AdS}_3$. If one attempts to deform the black hole solution to $\mathrm{AdS}_3$ by varying $M$, naked singularities are encountered between $M = 0$ and $M = -\frac{1}{8G}$. We will view the nonzero mass black holes as excitations of the vacuum described by the $M = 0$ black hole.

This raises the question of the role played by the $\mathrm{AdS}_3$ vacuum. A beautiful answer to this question was given in [16] which however requires supersymmetry. In a supersymmetric conformal field theory the Ramond ground state, which has periodic boundary conditions and is annihilated by the supercharge, has $M = 0$. We identify this with the zero-mass black hole (3.6). The Neveu-Schwarz ground state has antiperiodic boundary conditions and is not supersymmetric. It has a mass shift

$$L_0 = \bar{L}_0 = -\frac{c}{24}\,. \tag{3.7}$$

Using the formula (2.7) for $c$ and (3.4) for $M$ we find the energy of the Neveu-Schwarz ground state is

$$M = -\frac{1}{8G}\,, \tag{3.8}$$

which leads us to identify it with $\mathrm{AdS}_3$. Further evidence for this identification comes from the fact that the covariantly constant $\mathrm{AdS}_3$ spinors are antiperiodic under $\phi \to \phi + 2\pi$ (because it is a $2\pi$ rotation), while the covariantly constant spinors in the extremal $\ell M = J$ black hole geometries are periodic [16]. Additionally it can be seen in euclidean space [17] that the coordinate transformation relating (2.3) to (3.6) is exponential in $t$ and $\phi$, just like the map from the plane to the cylinder.

In a theory with local dynamics, a non-extremal BTZ black hole in empty space will Hawking radiate. However unlike the higher dimensional examples, there is (with appropriate boundary conditions at infinity) a stable, non-extremal endpoint corresponding to a black hole in thermal equilibrium with a radiation bath [18]. This is possible because an infinite radiation bath in $\mathrm{AdS}_3$ has finite energy due to an infinite temperature redshift at infinity. The generic hamiltonian eigenstate of the $\mathrm{AdS}_3$ conformal field theory presumably corresponds to such an equilibrium state.

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# 4. Other examples

The Bekenstein-Hawking entropy of a black hole depends only on the area of its horizon. In order to understand it only the near-horizon geometry of the black hole is relevant. Hence the considerations of this paper apply to any black hole whose near horizon geometry is $\mathrm{AdS}_3$ up to global identifications. There are many examples of this.

One example, considered in the string theory context in [10], is black strings in six dimension with charges $Q_1$, $Q_5$ and longitudinal momentum $n$. The near horizon geometry of this black string is locally $AdS_3 \times S^3 \times M^4$ with $M^4$ either $K^3$ or $T^4$ [19]. This may be regarded as quantum gravity on $\mathrm{AdS}_3$ with an infinite tower of matter fields from both massive string states and Kaluza-Klein modes of the $S^3 \times M^4$ compactification. Therefore the states are a representation of the Virasoro algebra (2.6).

The longitudinal direction along this black string lies within the $\mathrm{AdS}_3$. The black string can be periodically identified to give a black hole in five dimensions. The near-horizon geometry is then a BTZ black hole. For $n = 0$ one obtains the $M = J = 0$ black hole, while the generic black string yields the generic BTZ black hole.

An example involving four-dimensional extremal charged black holes is considered in [20], where the $U(1)$ arises from an internal circle. The near horizon geometry involves a $U(1)$ bundle over $\mathrm{AdS}_2$. The geometry of the bundle is a quotient of $\mathrm{AdS}_3$. The discrete identification group lies entirely within one of the two $SL(2,R)$s. Such geometries are considered in [21].

# 5. Microscopic derivation of the black hole entropy

We wish to count the number of excitations of the $\mathrm{AdS}_3$ vacuum with mass $M$ and angular momentum $J$ in the semiclassical regime of large $M$. According to (3.4) and (2.7) large $M$ implies

$$n_R + n_L \gg c\,, \tag{5.1}$$

where $n_R$ ($n_L$) is the eigenvalue of $L_0$ ($\bar{L}_0$). The asymptotic growth of the number states of a conformal field theory with central charge $c$ is then given by [14]

$$S = 2\pi\sqrt{\frac{cn_R}{6}} + 2\pi\sqrt{\frac{cn_L}{6}}\,. \tag{5.2}$$

Using (2.7), (3.4) and (3.5), this is

$$S = \pi\sqrt{\frac{\ell(\ell M + J)}{2G}} + \pi\sqrt{\frac{\ell(\ell M - J)}{2G}}\,, \tag{5.3}$$

in exact agreement with the Bekenstein-Hawking result (3.3) for the BTZ black hole.

## 6. Relation to previous derivations

The microscopic derivation of the previous section rests on a key assumption: the required $2+1$ quantum theory of gravity must exist. The details of the theory are not important, except in that the states must as discussed behave properly under diffeomorphisms. These are several constructions of such theories in which microscopic derivations of the entropy have previously been given. It is instructive to compare these with the present derivation.

The first is Carlip's derivation [7] of the entropy in pure $2+1$ gravity with no matter. This theory has no local degrees of freedom, and can be recast as a topological Chern-Simons theory [22, 23]. There are nevertheless boundary degrees of freedom at the black hole horizon which are described by a $1+1$ conformal field theory. Enumeration of the boundary states yields the Bekenstein-Hawking entropy. This boundary conformal field theory has a different central charge ($c \sim 6$ for large black holes) and so is not the same as the conformal field theory discussed here. Nevertheless it seems likely there is some connection between the approaches, which needs to be better understood. Relevant work in this direction appeared in [24] where it was shown that, for a boundary at infinity rather than the horizon, the Chern-Simons theory could be recast as a Liouville conformal field theory living on the boundary. An adaptation to string theory was discussed in [25].

The second example is the five dimensional string theory black hole with charges $Q_1$, $Q_5$ and $n$. It was indeed found in [10] that the black hole states are described by a $c = 6Q_1Q_5$ conformal field theory. This agrees with (2.7) since in the effective three-dimensional theory $\ell = 2\pi\alpha'\sqrt{g}(Q_1Q_5)^{1/4}/V^{1/4}$ and $G^{-1} = 2V^{1/4}(Q_1Q_5)^{3/4}/\pi\alpha'\sqrt{g}$.[4]

What is the relation between these two derivations of the black hole conformal field theory? The string theory black hole has two descriptions: one as a (string-corrected) supergravity solution, and the other as a bound state of $Q_1$ D-onebranes and $Q_5$ D-fivebranes with momentum $n$. For large $gQ$, but small string coupling $g$, string perturbation theory about the supergravity solution generally provides a good description. For small $gQ$, D-brane perturbation theory is generally good. It might appear there is no overlap in the regions of validity of the two descriptions. However, D-brane and supergravity descriptions have an overlapping region of validity for large $gQ$ in the near–horizon small $r$ region [26]. The supergravity picture is good because the horizon is a smooth place even though $r$ is small. The D-brane picture is also useful because small $r$ means low energies and higher dimension corrections to the D-brane gauge theory are suppressed. However, since $gQ$ is large, it is a large-$N$ gauge theory in this region.

This observation was made more precise in [13] (along with interesting generalizations to $\mathrm{AdS}_n$)with the introduction of a certain scaling limit (see also [19, 27, 25]). In

[4] $V$ is the four-volume associated with the compactification to six dimensions and we use units in which the ten-dimensional Newton's constant is given by $G = g^2 8\pi^6(\alpha')^4$.



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this limit, on the supergravity side, only the near-horizon theory of strings on $AdS_3$ remains, while on the D-brane side one has only the conformal field theory limit of the D-brane gauge theory. It was further noted [13] that the equivalence of these two theories was consistent with the global $SL(2,R)_L \otimes SL(2,R)_R$ symmetry of the ground state. With the $AdS_3$ boundary conditions defined in [12], one can go a step further and relate the action of the full *local* conformal group on both sides.

Of course, string theory enables one to go well beyond the considerations of this paper. For example one can not only determine the central charge of the theory (which is all that is needed for the entropy), but the exact conformal theory and degeneracies at every mass level.

## 7. Discussion

Where exactly are the states accounting for the black hole entropy? The states of the $AdS_3$ conformal field theory are associated to the $(t, \phi)$ cylinder, and not with any particular value of the radius $r$. The dynamics of the theory can be described in terms of evolution on this cylinder without introducing fields which depend on $r$. In this description nothing crosses the horizon, and there is no information loss.

The disappearance of $r$ in the description of the quantum theory is not surprising in the context of pure $2+1$ gravity, since that is a topological theory with only boundary degrees of freedom. However our discussion also applies to string theory in which one might expect new degrees of freedom at every value of $r$. This is a concrete example of the holographic principle advocated in [28, 29].

It would certainly be of interest to understand in greater detail the nature of the quantum states and dynamics in explicit examples. Consider the string theory black hole made from the compactification of $Q_5$ NS fivebranes and $Q_1$ fundamental strings. In this case the near-horizon string theory can be represented by a semi-conventional *worldsheet* conformal field theory: a level $Q_5$ $SL(2,R)$ WZW model, with discrete identifications from compactification [30, 20, 8]. The spectrum of this theory is not well-understood because it is noncompact. However perhaps it can be better organized utilizing the action of the full conformal group on the $SL(2,R)$=$AdS_3$ target space. The Hilbert space for pure $2+1$ gravity also needs to be better understood, perhaps using the representation as a Liouville theory [24].

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