

INTRODUCTION TO LOOP QUANTUM GRAVITY AND SPIN FOAMS

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These notes are a didactic overview of the non perturbative and background independent approach to a quantum theory of gravity known as loop quantum gravity. The definition of real connection variables for general relativity, used as a starting point in the program, is described in a simple manner. The main ideas leading to the definition of the quantum theory are naturally introduced and the basic mathematics involved is described. The main predictions of the theory such as the discovery of Planck scale discreteness of geometry and the computation of black hole entropy are reviewed. The quantization and solution of the constraints is explained by drawing analogies with simpler systems. Difficulties associated with the quantization of the scalar constraint are discussed.

In a second part of the notes, the basic ideas behind the spin foam approach are presented in detail for the simple solvable case of 2+1 gravity. Some results and ideas for four dimensional spin foams are reviewed.

Keywords: Quantum Gravity; Loop variables; Non-perturbative methods; Path Integrals; Yang-Mills theory.

1. Introduction: why non perturbative quantum gravity?

The remarkable experimental success of the standard model in the description of fundamental interactions is the greatest achievement of (relativistic) quantum field theory. Standard quantum field theory provides an accomplished unification between the principles of quantum mechanics and special relativity. The standard model is a very useful example of a quantum field theory on the fixed background geometry of Minkowski spacetime. As such, the standard model can only be regarded as an approximation of the description of fundamental interactions valid when the gravitational field is negligible. The standard model is indeed a very good approximation describing particle physics in the lab and in a variety of astrophysical situations because of the weakness of the gravitational force at the scales of interest (see for instance the lecture by Rogério Rosenfeld¹). Using the techniques of quantum field theory on curved spacetimes² one might hope to extend the applicability

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of the standard model to situations where a non trivial, but weak, gravitational field is present. These situations are thought to be those where the spacetime curvature is small in comparison with the Planck scale, although a clear justification for its regime of validity in strong gravitational fields seems only possible when a full theory of quantum gravity is available. Having said this, there is a number of important physical situations where we do not have any tools to answer even the simplest questions. In particular classical general relativity predicts the existence of singularities in physically realistic situations such as those dealing with black hole physics and cosmology. Near spacetime singularities the classical description of the gravitational degrees of freedom simply breaks down. Questions related to the fate of singularities in black holes or in cosmological situations as well as those related with apparent information paradoxes are some of the reasons why we need a theory of quantum gravity. This new theoretical framework—yet to be put forward—aims at a consistent description unifying or, perhaps more appropriately, underlying the principles of general relativity and quantum mechanics.

The gravitational interaction is fundamentally different from all the other known forces. The main lesson of general relativity is that the degrees of freedom of the gravitational field are encoded in the geometry of spacetime. The spacetime geometry is fully dynamical: in gravitational physics the notion of absolute space on top of which ‘things happen’ ceases to make sense. The gravitational field defines the geometry on top of which its own degrees of freedom and those of matter fields propagate. This is clear from the perspective of the initial value formulation of general relativity, where, given suitable initial conditions on a 3-dimensional manifold, Einstein’s equations determine the dynamics that ultimately allows for the reconstruction of the spacetime geometry with all the matter fields propagating on it. A spacetime notion can only be recovered *a posteriori* once the complete dynamics of the coupled geometry-matter system is worked out. Matter affects the dynamics of the gravitational field and is affected by it through the non trivial geometry that the latter defines^a. General relativity is not a theory of fields moving on a curved background geometry; general relativity is a theory of fields moving on top of each other⁴.

In classical physics general relativity is not just a successful description of the nature of the gravitational interaction. As a result of implementing the principles of general covariance, general relativity provides the basic framework to assessing the physical world by cutting all ties to concepts of absolute space. It represents the result of the long-line of developments that go all the way back to the thought experiments of Galileo about the relativity of the motion, to the arguments of Mach about the nature of space and time, and finally to the magnificent conceptual synthesis of Einstein’s: the world is relational. There is no well defined notion of absolute space and it only makes sense to describe physical entities in relation to

^aSee for instance Theorem 10.2.2 in ³.

other physical entities. This conceptual viewpoint is fully represented by the way matter and geometry play together in general relativity. The full consequences of this in quantum physics are yet to be unveiled^b.

When we analyze a physical situation in pre-general-relativistic physics we separate what we call the system from the relations to other objects that we call the reference frame. The spacetime geometry that we describe with the aid of coordinates and a metric is a mathematical idealization of what in practice we measure using rods and clocks. Any meaningful statement about the physics of the system is a statement about the relation of some degrees of freedom in the system with those of what we call the frame. The key point is that when the gravitational field is trivial there exist a preferred set of physical systems whose dynamics is very simple. These systems are *inertial observers*; for instance one can think of them as given by a spacial grid of clocks synchronized by the exchange of light signals. These physical objects provide the definition of inertial coordinates and their mutual relations can be described by a Minkowski metric. As a result in pre-general relativistic physics we tend to forget about rods and clocks that define (inertial) frames and we talk about time t and position x and distances measure using the flat metric η_{ab} ^c. However, what we are really doing is comparing the degrees of freedom of our system with those of a space grid of world-lines of physical systems called *inertial observers*. From this perspective, statements in special relativity are in fact diffeomorphism invariant. The physics from the point of view of an experimentalist—dealing with the system itself, clocks, rods, and their mutual relations—is completely independent of coordinates. In general relativity this property of the world is confronted head on: only relational (coordinate-independent, or diffeomorphism invariant) statements are meaningful (see discussion about the hole argument in ⁴). There are no simple family of observers to define physical coordinates as in the flat case, so we use arbitrary labels (coordinates) and require the physics to be independent of them.

In this sense, the principles of general relativity state some basic truth about the nature of the classical world. The far reaching consequences of this in the quantum realm are certainly not yet well understood.^d However, it is very difficult to imagine that a notion of absolute space would be saved in the next step of development of our understanding of fundamental physics. Trying to build a theory of quantum gravity based on a notion of background geometry would be, from this perspective, reminiscent of the efforts by contemporaries of Copernicus of describing planetary motion in terms of the geocentric framework.

^bFor a fascinating account of the conceptual subtleties of general relativity see Rovelli's book⁴.

^cIn principle we first use rods and clocks to realize that in the situation of interests (e.g. an experiment at CERN) the gravitational field is trivial and then we just encode this information in a fix background geometry: Minkowski spacetime.

^dA reformulation of quantum mechanics from a relational perspective has been introduced by Rovelli⁵ and further investigated in the context of conceptual puzzles of quantum mechanics^{6,7,8,9,10}. It is also possible that a radical change in the paradigms of quantum mechanics is necessary in the conceptual unification of gravity and the quantum^{11,12,13,14}.

1.1. *Perturbative quantum gravity*

Let us make some observations about the problems of standard perturbative quantum gravity. In doing so we will revisit the general discussion above, in a special situation. In standard perturbative approaches to quantum gravity one attempts to describe the gravitational interaction using the same techniques applied to the definition of the standard model. As these techniques require a notion of non dynamical background one (arbitrarily) separates the degrees of freedom of the gravitational field in terms of a background geometry η_{ab} for $a, b = 1 \cdots 4$ —fixed once and for all—and dynamical metric fluctuations h_{ab} . Explicitly, one writes the spacetime metric as

$$g_{ab} = \eta_{ab} + h_{ab}. \quad (1)$$

Notice that the previous separation of degrees of freedom has no intrinsic meaning in general relativity. In other words, for a generic space time metric g_{ab} we can write

$$g_{ab} = \eta_{ab} + h_{ab} = \tilde{\eta}_{ab} + \tilde{h}_{ab}, \quad (2)$$

where η_{ab} and $\tilde{\eta}_{ab}$ can lead to different background light-cone structures of the underlying spacetime (M, g_{ab}) ; equally natural choices of flat background metrics lead to different Minkowski metrics in this sense. This is quite dangerous if we want to give any physical meaning to the background, e.g., the light cone structures of the two ‘natural’ backgrounds will be generally different providing different notions of causality! Equation (1) is used in the classical theory in very special situations when one considers perturbations of a given background η_{ab} . In quantum gravity one has to deal with arbitrary superpositions of spacetimes; the above splitting can at best be meaningful for very special semi-classical states ‘peaked’, so to say, around the classical geometry η_{ab} with small fluctuations. It is very difficult to imagine how such a splitting can be useful in considering general states with arbitrary quantum excitations at all scales. Specially because of the dual role of the gravitational field that simultaneously describes the geometry and its own dynamical degrees of freedom. More explicitly, in the standard background dependent quantization the existence of a fixed background geometry is fundamental in the definition of the theory. For instance, one expects fields at space-like separated points to commute alluding to standard causality considerations. Even when this is certainly justified in the range of applicability of the standard model, in a background dependent quantization of gravity one would be using the causal structure provided by the unphysical background η_{ab} . Yet we know that the notion of causality the world really follows is that of the full g_{ab} (see Footnote a). This difficulty has been raised several times (see for instance³). Equation (1) could be meaningful in special situations dealing with semi-classical issues, but it does not seem to be of much use if one wants to describe the fundamental degrees of freedom of quantum gravity.

If we ignore all these issues and try to setup a naive perturbative quantization of gravity we find that the theory is non renormalizable. This can be expected from

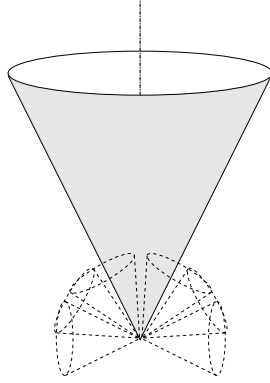


Fig. 1. The larger cone represents the light-cone at a point according to the ad hoc background η_{ab} . The smaller cones are a cartoon representation of the fluctuations of the true gravitational field represented by g_{ab} .

dimensional analysis as the quantity playing the role of the coupling constant turns out to be the Planck length ℓ_p . The non renormalizability of perturbative gravity is often explained through an analogy with the (non-renormalizable) Fermi's four fermion effective description of the weak interaction¹⁵. Fermi's four fermion theory is known to be an effective description of the (renormalizable) Weinberg-Salam theory. The non renormalizable UV behavior of Fermi's four fermion interaction is a consequence of neglecting the degrees of freedom of the exchanged massive gauge bosons which are otherwise disguised as the dimension-full coupling $\Lambda_{\text{Fermi}} \approx 1/m_W^2$ at momentum transfer much lower than the mass of the W particle ($q^2 \ll m_W^2$). A similar view is applied to gravity to promote the search of a more fundamental theory which is renormalizable or finite (in the perturbative sense) and reduces to general relativity at low energies. From this perspective it is argued that the quantization of general relativity is a hopeless attempt to quantizing a theory that does not contain the fundamental degrees of freedom.

These arguments, based on background dependent concepts, seem at the very least questionable in the case of gravity. Although one should expect the notion of a background geometry to be useful in certain semi-classical situations, the assumption that such structure exists all the way down to the Planck scale is inconsistent with what we know about gravity and quantum mechanics. General considerations indicate that standard notions of space and time are expected to fail near the Planck scale ℓ_p ^e. From this viewpoint the non renormalizability of perturbative quantum

^eFor instance a typical example is to use a photon to measure distance. The energy of the photon in our lab frame is given by $E_\gamma = hc/\lambda$. We put the photon in a cavity and produce a standing wave measuring the dimensions of the cavity in units of the wavelength. The best possible precision is attained when the Schwarzschild radius corresponding to energy of the photon is of the order of its wavelength. Beyond that the photon can collapse to form a black hole around some of the maxima of the standing wave. This happens for a value λ_c for which $\lambda_c \approx GE_\gamma/c^4 = hG/(\lambda_c c^3)$. The solution is $\lambda_c \approx \sqrt{hG/c^3}$ which is Planck length.

gravity is indicative of the inconsistency of the separation of degrees of freedom in (1). The nature of spacetime is expected to be very different from the classical notion in quantum gravity. The treatment that uses (1) as the starting point is assuming a well defined notion of background geometry at all scales which directly contradicts these considerations.

In fact one could read the issues of divergences in an alternative way. The accepted jargon is that in a renormalizable theory the microscopic physics does not affect low energy processes as its effects can be recast in simple redefinition of a finite set of parameters, i.e., renormalization. In the case of gravity this miracle does not happen. All correlation functions blow up and one would need to add an infinite number of corrections to the original Lagrangian when integrating out microscopic physical degrees of freedom. Now imagine for the moment that a non perturbative quantization of general relativity was available. If we now want to recover a ‘low energy effective’ description of a physical situation—where (low energy) physical considerations single out a preferred background—we would expect the need of increasingly higher derivative correction terms to compensate the fact the ‘high energy’^f degrees of freedom know nothing about that low energy background. In this sense the objections raised above to the violent and un-natural splitting (1) are related to the issue of non renormalizability of general relativity. The nature of the gravitational interaction is telling us that the standard paradigm of renormalization (based on the notion of a fixed background) is no longer applicable. A reformulation of quantum field theory is necessary to cope with background independence.

It is possible that new degrees of freedom would become important at more fundamental scales. After all, that is the story of the path that lead to the standard model where higher energy experiments has often lead to the discovery of new interactions. It is also possible that including these degrees of freedom might be very important for the consistency of the theory of quantum gravity. However, there is constraint that seem hardly avoidable: if we want to get a quantum theory that reproduces gravity in the semi-classical limit we should have a background independent formalism. In loop quantum gravity one stresses this viewpoint. The hope is that learning how to define quantum field theory in the absence of a background is a key ingredient in a recipe for quantum gravity. Loop quantum gravity uses the quantization of general relativity as a playground to achieve this goal.

I would like to finish this subsection with a quote of Weinberg’s 1980 paper that would serve as introduction for the next section:

“It is possible that this problem—the non renormalizability of general relativity—

^fThe notion of energy is observer dependent in special relativity. In the background independent context is not even defined. We use the terminology ‘high (low) energy’ in this introduction to refer to the fundamental (semi-classical) degrees of freedom in quantum gravity. However, the reader should not take this too literally. In fact if one tries to identify the fundamental degrees of freedom of quantum gravity with a preferred observer conflict with observation seem inevitable^{16,17,18}.

has arisen because the usual flat-space formalism of quantum field theory simply cannot be applied to gravitation. After all, gravitation is a very special phenomenon, involving as it does the very topology of space and time”

All these considerations make the case for a background independent approach to quantum gravity. The challenge is to define quantum field theory in the absence of any preestablished notion of distance: quantum field theory without a metric.

1.2. Loop quantum gravity

Loop quantum gravity is an attempt to define a quantization of gravity paying special attention to the conceptual lessons of general relativity (the reader is encouraged to read Rovelli’s book: *Quantum Gravity*⁴ as well as the recent review by Ashtekar and Lewandowski¹⁹. For a detailed account of the mathematical techniques involved in the construction of the theory see Thiemann’s book²⁰). The theory is explicitly formulated in a background independent, and therefore, non perturbative fashion. The theory is based on the Hamiltonian (or canonical) quantization of general relativity in terms of variables that are different from the standard metric variables. In terms of these variables general relativity is cast into the form of a background independent $SU(2)$ gauge theory partly analogous to $SU(2)$ Yang-Mills theory. The main prediction of loop quantum gravity (LQG) is the discreteness^{21,22} of the spectrum of geometrical operators such as area and volume. The discreteness becomes important at the Planck scale while the spectrum of geometric operators crowds very rapidly at ‘low energy scales’ (large geometries). This property of the spectrum of geometric operators is consistent with the smooth spacetime picture of classical general relativity.

Thus it is not surprising that perturbative approaches would lead to inconsistencies. In splitting the gravitational field degrees of freedom as in (1) one is assuming the existence of a background geometry which is smooth all the way down to the Planck scale. As we consider contributions from ‘higher energies’, this assumption is increasingly inconsistent with the fundamental structure discovered in the non perturbative treatment.

The theory is formulated in four dimensions. Matter can be coupled to the gravitational degrees of freedom and the inclusion of matter has been studied in great detail. However, throughout these lectures we will study the pure gravitational sector in order to simplify the presentation. On the super-symmetric extension of canonical loop quantum gravity see ^{23,24,25,26,27}. A spin foam model for 3d super-gravity has been defined by Livine and Oeckl²⁸.

Despite the achievements of LQG there remain important issues to be addressed. There are difficulties of a rather technical nature related to the complete characterization of dynamics and the quantization of the so-called scalar constraint. We will review these once we have introduced the basic formalism. The most important question, however, remains open: whether the semi-classical limit of the LQG is

consistent with general relativity and the Standard Model.

Before starting with the more technical material let us say a few words regarding *unification*. In loop quantum gravity matter is essentially added by coupling general relativity with it at the classical level and then performing the (background independent) canonical quantization. At this stage of development of the approach it is not clear if there is some restriction in the kind of interactions allowed or if degrees of freedom corresponding to matter could arise in some natural way directly from those of geometry. However, there is a unification which is addressed head on by LQG: the need to find a common framework in which to describe quantum field theories in the absence of an underlying spacetime geometry. The implications of this are far from being fully developed.

2. Hamiltonian formulation of general relativity: the new variables

In this section we introduce the variables that are used in the definition of loop quantum gravity. We will present the $SU(2)$ Ashtekar-Barbero variables by starting with conventional ADM variables, introducing triads at the canonical level, and finally performing the well known canonical transformation that leads to the new variables. This is, in my opinion, the simplest way to get to the new variables. For a derivation that emphasizes the covariant four dimensional character of these variables see ¹⁹.

2.1. Canonical analysis in ADM variables

The action of general relativity in metric variables is given by the Einstein-Hilbert action

$$I[g_{\mu\nu}] = \frac{1}{2\kappa} \int dx^4 \sqrt{-g} R, \quad (3)$$

where $\kappa = 8\pi G/c^3 = 8\pi\ell_p^2/\hbar$, g is the determinant of the metric g_{ab} and R is the Ricci scalar. The details of the Hamiltonian formulation of this action in terms of ADM variables can be found in³. One introduces a foliation of spacetime in terms of space-like three dimensional surfaces Σ . For simplicity we assume Σ has no boundaries. The ten components of the spacetime metric are replaced by the six components of the induced Riemannian metric q_{ab} of Σ plus the three components of the shift vector N_a and the lapse function N . In terms of these variables, after performing the standard Legendre transformation, the action of general relativity becomes

$$I[q_{ab}, \pi^{ab}, N_a, N] = \frac{1}{2\kappa} \int dt \int_{\Sigma} dx^3 [\pi^{ab} \dot{q}_{ab} + 2N_b \nabla^{(3)}{}_a (q^{-1/2} \pi^{ab}) + N (q^{1/2} [R^{(3)} - q^{-1} \pi_{cd} \pi^{cd} + \frac{1}{2} q^{-1} \pi^2])], \quad (4)$$

where π^{ab} are the momenta canonically conjugate to the space metric q_{ab} , $\pi = \pi^{ab} q_{ab}$, $\nabla^{(3)}{}_a$ is the covariant derivative compatible with the metric q_{ab} , q is the

determinant of the space metric and $R^{(3)}$ is the Ricci tensor of q_{ab} . The momenta π^{ab} are related to the extrinsic curvature K_{ab}^g of Σ by

$$\pi^{ab} = q^{-1/2}(K^{ab} - Kq^{ab}) \quad (5)$$

where $K = K_{ab}q^{ab}$ and indices are raised with q^{ab} . Variations with respect to the lapse and shift produce the four constraint equations:

$$-V^b(q_{ab}, \pi^{ab}) = 2\nabla^{(3)}{}_a(q^{-1/2}\pi^{ab}) = 0, \quad (6)$$

and

$$-S(q_{ab}, \pi^{ab}) = (q^{1/2}[R^{(3)} - q^{-1}\pi_{cd}\pi^{cd} + 1/2q^{-1}\pi^2]) = 0. \quad (7)$$

$V^b(q_{ab}, \pi^{ab})$ is the so-called vector constraint and $S(q_{ab}, \pi^{ab})$ is the scalar constraint. With this notation the action can be written as

$$I[q_{ab}, \pi^{ab}, N_a, N] = \frac{1}{2\kappa} \int dt \int_{\Sigma} dx^3 [\pi^{ab}\dot{q}_{ab} - N_b V^b(q_{ab}, \pi^{ab}) - NS(q_{ab}, \pi^{ab})], \quad (8)$$

where we identify the Hamiltonian density $\mathcal{H}(q_{ab}, \pi^{ab}, N_a, N) = N_b V^b(q_{ab}, \pi^{ab}) + NS(q_{ab}, \pi^{ab})$. The Hamiltonian is a linear combination of (first class) constraints, i.e., it vanishes identically on solutions of the equations of motion. This is a generic property of generally covariant systems. The symplectic structure can be read off the previous equations, namely

$$\{\pi^{ab}(x), q_{cd}(y)\} = 2\kappa \delta_{(c}^a \delta_{d)}^b \delta(x, y), \quad \{\pi^{ab}(x), \pi^{cd}(y)\} = \{q_{ab}(x), q_{cd}(y)\} = 0 \quad (9)$$

There are six configuration variables q_{ab} and four constraint equations (6) and (7) which implies the two physical degrees of freedom of gravity ^h.

2.2. Toward the new variables: the triad formulation

Now we do a very simple change of variables. The idea is to use a triad (a set of three 1-forms defining a frame at each point in Σ) in terms of which the metric q_{ab} becomes

$$q_{ab} = e_a^i e_b^j \delta_{ij}, \quad (10)$$

where $i, j = 1, 2, 3$. Using these variables we introduce the densitized triad

$$E_i^a := \frac{1}{2} \epsilon^{abc} \epsilon_{ijk} e_b^j e_c^k. \quad (11)$$

Using this definition, the inverse metric q^{ab} can be related to the densitized triad as follows

$$qq^{ab} = E_i^a E_j^b \delta^{ij}. \quad (12)$$

^gThe extrinsic curvature is given by $K_{ab} = \frac{1}{2} \mathcal{L}_n q_{ab}$ where n^a is the unit normal to Σ .

^hThis counting of physical degrees of freedom is correct because the constraints are first class²⁹.

We also define

$$K_a^i := \frac{1}{\sqrt{\det(E)}} K_{ab} E_j^b \delta^{ij}. \quad (13)$$

A simple exercise shows that one can write the canonical term in (8) as

$$\pi^{ab} \dot{q}_{ab} = -\pi_{ab} \dot{q}^{ab} = 2E_a^i \dot{K}_a^i \quad (14)$$

and that the constraints $V^a(q_{ab}, \pi^{ab})$ and $S(q_{ab}, \pi^{ab})$ can respectively be written as $V^a(E_i^a, K_a^i)$ and $S(E_i^a, K_a^i)$. Therefore we can rewrite (8) in terms of the new variables. However, the new variables are certainly redundant, in fact we are using the nine E_i^a to describe the six components of q^{ab} . The redundancy has a clear geometrical interpretation: the extra three degrees of freedom in the triad correspond to our ability to choose different local frames e_a^i by local $SO(3)$ rotations acting in the internal indices $i = 1, 2, 3$. There must then be an additional constraint in terms of the new variables that makes this redundancy manifest. The missing constraint comes from (13): we overlooked the fact that $K_{ab} = K_{ba}$ or simply that $K_{[ab]} = 0$. By inverting the definitions (11) and (13) in order to write K_{ab} in terms of E_i^a and K_a^i one can show that the condition $K_{[ab]} = 0$ reduces to

$$G_i(E_j^a, K_a^j) := \epsilon_{ijk} E^{aj} K_a^k = 0. \quad (15)$$

Therefore we must include this additional constraint to (8) if we want to use the new triad variables. With all this the action of general relativity becomes

$$I[E_j^a, K_a^j, N_a, N, N^j] = \frac{1}{\kappa} \int dt \int_{\Sigma} dx^3 \left[E_i^a \dot{K}_a^i - N_b V^b(E_j^a, K_a^j) - NS(E_j^a, K_a^j) - N^i G_i(E_j^a, K_a^j) \right], \quad (16)$$

where the explicit form of the constraints in terms of triad variables can be worked out from the definitions above. The reader is encouraged to do this exercise but it is not going to be essential to understanding what follows (expressions for the constraints can be found in reference³⁰). The symplectic structure now becomes

$$\{E_j^a(x), K_b^i(y)\} = \kappa \delta_b^a \delta_j^i \delta(x, y), \quad \{E_j^a(x), E_i^b(y)\} = \{K_a^j(x), K_b^i(y)\} = 0 \quad (17)$$

The counting of physical degrees of freedom can be done as before.

2.3. *New variables: the Ashtekar-Barbero connection variables*

The densitized triad (11) transforms in the vector representation of $SO(3)$ under redefinition of the triad (10). Consequently, so does its conjugate momentum K_a^i (see equation (13)). There is a natural $so(3)$ -connection that defines the notion of covariant derivative compatible with the triad. This connection is the so-called *spin connection* Γ_a^i and is characterized as the solution of Cartan's structure equations

$$\partial_{[a} e_{b]}^i + \epsilon^i{}_{jk} \Gamma_{[a}^j e_{b]}^k = 0 \quad (18)$$

The solution to the previous equation can be written explicitly in terms of the triad components

$$\Gamma_a^i = -\frac{1}{2}\epsilon^{ij}{}_{k}e_j^b \left(\partial_{[a}e_{b]}^k + \delta^{kl}\delta_{ms}e_l^c e_a^m \partial_b e_c^s \right), \quad (19)$$

where e_i^a is the inverse triad ($e_i^a e_a^j = \delta_i^j$). We can obtain an explicit function of the densitized triad— $\Gamma_a^i(E_j^b)$ —inverting (11) from where

$$e_a^i = \frac{1}{2} \frac{\epsilon_{abc}\epsilon^{ijk}E_j^b E_k^c}{\sqrt{|\det(E)|}} \quad \text{and} \quad e_i^a = \frac{\text{sgn}(\det(E)) E_i^a}{\sqrt{|\det(E)|}}. \quad (20)$$

The spin connection is an $so(3)$ connection that transforms in the standard inhomogeneous way under local $SO(3)$ transformations. The Ashtekar-Barbero^{31,32,33} variables are defined by the introduction of a new connection A_a^i given by

$$A_a^i = \Gamma_a^i + \gamma K_a^i, \quad (21)$$

where γ is any non vanishing real number called the Immirzi parameter³⁴. The new variable is also an $so(3)$ connection as adding a quantity that transforms as a vector to a connection gives a new connection. The remarkable fact about this new variable is that it is in fact conjugate to E_a^i . More precisely the Poisson brackets of the new variables are

$$\{E_j^a(x), A_b^i(y)\} = \kappa \gamma \delta_b^a \delta_j^i \delta(x, y), \quad \{E_j^a(x), E_i^b(y)\} = \{A_a^j(x), A_b^i(y)\} = 0. \quad (22)$$

All the previous equations follow trivially from (17) except for $\{A_a^j(x), A_b^i(y)\} = 0$ which requires more calculations. The reader is invited to check it.

Using the connection variables the action becomes

$$I[E_j^a, A_a^j, N_a, N, N^j] = \frac{1}{\kappa} \int dt \int_{\Sigma} dx^3 \left[E_i^a \dot{A}_i^a - N^b V_b(E_j^a, A_a^j) - NS(E_j^a, A_a^j) - N^i G_i(E_j^a, A_a^j) \right], \quad (23)$$

where the constraints are explicitly given by:

$$V_b(E_j^a, A_a^j) = E_j^a F_{ab} - (1 + \gamma^2) K_a^i G_i \quad (24)$$

$$S(E_j^a, A_a^j) = \frac{E_i^a E_j^b}{\sqrt{\det(E)}} \left(\epsilon^{ij}{}_{k} F_{ab}^k - 2(1 + \gamma^2) K_{[a}^i K_{b]}^j \right) \quad (25)$$

$$G_i(E_j^a, A_a^j) = D_a E_i^a, \quad (26)$$

where $F_{ab} = \partial_a A_b^i - \partial_b A_a^i + \epsilon^i{}_{jk} A_a^j A_b^k$ is the curvature of the connection A_a^i and $D_a E_i^a = \partial_a E_i^a + \epsilon_{ij}{}^k A_a^j E_k^i$ is the covariant divergence of the densitized triad. We have seven (first class) constraints for the 18 phase space variables (A_a^i, E_j^b). In addition to imposing conditions among the canonical variables, first class constraints are generating functionals of (infinitesimal) gauge transformations. From the 18-dimensional phase space of general relativity we end up with 11 fields necessary

to coordinatize the constraint surface on which the above seven conditions hold. On that 11-dimensional constraint surface, the above constraint generate a seven-parameter-family of gauge transformations. The reduce phase space is four dimensional and therefore the resulting number of physical degrees of freedom is *two*, as expected.

The constraint (26) coincides with the standard Gauss law of Yang-Mills theory (e.g. $\vec{\nabla} \cdot \vec{E} = 0$ in electromagnetism). In fact if we ignore (24) and (25) the phase space variables (A_a^i, E_j^b) together with the Gauss law (26) characterize the physical phase space of an $SU(2)^i$ Yang-Mills (YM) theory. The gauge field is given by the connection A_a^i and its conjugate momentum is the electric field E_j^b . Yang-Mills theory is a theory defined on a background spacetime geometry. Dynamics in such a theory is described by a non vanishing Hamiltonian—the Hamiltonian density of YM theory being $\mathcal{H} = E_a^i E_i^a + B_a^i B_i^a$. General relativity is a generally covariant theory and coordinate time plays no physical role. The Hamiltonian is a linear combination of constraints.^j Dynamics is encoded in the constraint equations (24),(25), and (26). In this sense we can regard general relativity in the new variables as a background independent relative of $SU(2)$ Yang-Mills theory. We will see in the sequel that the close similarity between these theories will allow for the implementation of techniques that are very natural in the context of YM theory.

2.3.1. Gauge transformations

Now let us analyze the structure of the gauge transformations generated by the constraints (24),(25), and (26). From the previous paragraph it should not be surprising that the Gauss law (26) generates local $SU(2)$ transformations as in the case of YM theory. Explicitly, if we define the smeared version of (26) as

$$G(\alpha) = \int_{\Sigma} dx^3 \alpha^i G_i(A_a^i, E_i^a) = \int_{\Sigma} dx^3 \alpha^i D_a E_i^a, \quad (27)$$

a direct calculation implies

$$\delta_G A_a^i = \{A_a^i, G(\alpha)\} = -D_a \alpha^i \quad \text{and} \quad \delta_G E_i^a = \{E_i^a, G(\alpha)\} = [E, \alpha]_i. \quad (28)$$

ⁱThe constraint structure does not distinguish $SO(3)$ from $SU(2)$ as both groups have the same Lie algebra. From now on we choose to work with the more fundamental (universal covering) group $SU(2)$. In fact this choice is physically motivated as $SU(2)$ is the gauge group if we want to include fermionic matter^{35,36,37}.

^jIn the physics of the standard model we are used to identifying the coordinate t with the physical time of a suitable family of observers. In the general covariant context of gravitational physics the coordinate time t plays the role of a label with no physical relevance. One can arbitrarily change the way we coordinatize spacetime without affecting the physics. This redundancy in the description of the physics (gauge symmetry) induces the appearance of constraints in the canonical formulation. The constraints in turn are the generating functions of these gauge symmetries. The Hamiltonian generates evolution in coordinate time t but because redefinition of t is pure gauge, the Hamiltonian is a constraint itself, i.e. $\mathcal{H} = 0$ on shell^{38,29}. More on this in the next section.

If we write $A_a = A_a^i \tau_i \in su(2)$ and $E^a = E_i^a \tau^i \in su(2)$, where τ_i are generators of $SU(2)$, we can write the finite version of the previous transformation

$$A'_a = gA_ag^{-1} + g\partial_ag^{-1} \quad \text{and} \quad E'^a = gE^ag^{-1}, \quad (29)$$

which is the standard way the connection and the electric field transform under gauge transformations in YM theory.

The vector constraint (24) generates three dimensional diffeomorphisms of Σ . This is clear from the action of the smeared constraint

$$V(N^a) = \int_{\Sigma} dx^3 N^a V_a(A_a^i, E_i^a) \quad (30)$$

on the canonical variables

$$\delta_V A_a^i = \{A_a^i, V(N^a)\} = \mathcal{L}_N A_a^i \quad \text{and} \quad \delta_V E_i^a = \{E_i^a, V(N^a)\} = \mathcal{L}_N E_i^a, \quad (31)$$

where \mathcal{L}_N denotes the Lie derivative in the N^a direction. The exponentiation of these infinitesimal transformations leads to the action of finite diffeomorphisms on Σ .

Finally, the scalar constraint (25) generates coordinate time evolution (up to space diffeomorphisms and local $SU(2)$ transformations). The total Hamiltonian $H[\alpha, N^a, N]$ of general relativity can be written as

$$H(\alpha, N^a, N) = G(\alpha) + V(N^a) + S(N), \quad (32)$$

where

$$S(N) = \int_{\Sigma} dx^3 NS(A_a^i, E_i^a). \quad (33)$$

Hamilton's equations of motion are therefore

$$\dot{A}_a^i = \{A_a^i, H(\alpha, N^a, N)\} = \{A_a^i, S(N)\} + \{A_a^i, G(\alpha)\} + \{A_a^i, V(N^a)\}, \quad (34)$$

and

$$\dot{E}_i^a = \{E_i^a, H(\alpha, N^a, N)\} = \{E_i^a, S(N)\} + \{E_i^a, G(\alpha)\} + \{E_i^a, V(N^a)\}. \quad (35)$$

The previous equations define the action of $S(N)$ up to infinitesimal $SU(2)$ and diffeomorphism transformations given by the last two terms and the values of α and N^a respectively. In general relativity coordinate time evolution does not have any physical meaning. It is analogous to a $U(1)$ gauge transformation in QED.

2.3.2. Constraints algebra

Here we simply present the structure of the constraint algebra of general relativity in the new variables.

$$\{G(\alpha), G(\beta)\} = G([\alpha, \beta]), \quad (36)$$

where $\alpha = \alpha^i \tau_i \in su(2)$, $\beta = \beta^i \tau_i \in su(2)$ and $[\alpha, \beta]$ is the commutator in $su(2)$.

$$\{G(\alpha), V(N^a)\} = -G(\mathcal{L}_N \alpha). \quad (37)$$

$$\{G(\alpha), S(N)\} = 0. \quad (38)$$

$$\{V(N^a), V(M^a)\} = V([N, M]^a), \quad (39)$$

where $[N, M]^a = N^b \partial_b M^a - M^b \partial_b N^a$ is the vector field commutator.

$$\{S(N), V(N^a)\} = -S(\mathcal{L}_N N). \quad (40)$$

Finally

$$\{S(N), S(M)\} = V(S^a) + \text{terms proportional to the Gauss constraint}, \quad (41)$$

where for simplicity we are ignoring the terms proportional to the Gauss law (the complete expression can be found in ¹⁹) and

$$S^a = \frac{E_i^a E_j^b \delta^{ij}}{|\det E|} (N \partial_b M - M \partial_b N). \quad (42)$$

Notice that instead of structure constants, the r.h.s. of (41) is written in terms of field dependent structure functions. For this reason it is said that the constraint algebra does not close in the BRS sense.

2.3.3. *Ashtekar variables*

The connection variables introduced in this section do not have a simple relationship with four dimensional fields. In particular the connection (21) cannot be obtained as the pullback to Σ of a spacetime connection³⁹. Another observation is that the constraints (24) and (25) dramatically simplify when $\gamma^2 = -1$. Explicitly, for $\gamma = i$ we have

$$V_b^{SD} = E_j^a F_{ab} \quad (43)$$

$$S^{SD} = \frac{E_i^a E_j^b}{\sqrt{\det(E)}} \epsilon^{ij}{}_k F_{ab}^k \quad (44)$$

$$G_i^{SD} = D_a E_i^a, \quad (45)$$

where SD stands for self dual; a notation that will become clear below. Notice that with $\gamma = i$ the connection (21) is complex³⁰ (i.e. $A_a \in sl(2, \mathbb{C})$). To recover real general relativity these variables must be supplemented with the so-called reality condition that follows from (21), namely

$$A_a^i + \bar{A}_a^i = \Gamma_a^i(E). \quad (46)$$

In addition to the simplification of the constraints, the connection obtained for this choice of the Immirzi parameter is simply related to a spacetime connection. More precisely, it can be shown that A_a is the pullback of ω_μ^{IJ} ($I, J = 1, \dots, 4$) where

$$\omega_\mu^{IJ} = \frac{1}{2}(\omega_\mu^{IJ} - \frac{i}{2}\epsilon^{IJ}_{KL}\omega_\mu^{KL}) \quad (47)$$

is the self dual part of a Lorentz connection ω_μ^{IJ} . The gauge group—generated by the (complexified) Gauss constraint—is in this case $SL(2, \mathbb{C})$.

Loop quantum gravity was initially formulated in terms of these variables. However, there are technical difficulties in defining the quantum theory when the connection is valued in the Lie algebra of a non compact group. Progress has been achieved constructing the quantum theory in terms of the real variables introduced in Section 2.3.

2.4. Geometric interpretation of the new variables

The geometric interpretation of the connection A_a^i , defined in (21), is standard. The connection provides a definition of *parallel transport* of $SU(2)$ spinors on the space manifold Σ . The natural object is the $SU(2)$ element defining parallel transport along a path $e \subset \Sigma$ also called *holonomy* denoted $h_e[A]$, or more explicitly

$$h_e[A] = P \exp - \int_e A, \quad (48)$$

where P denotes a path-order-exponential (more details in the next section).

The densitized triad—or electric field— E_i^a also has a simple geometrical meaning. E_i^a encodes the full background independent Riemannian geometry of Σ as is clear from (12). Therefore, any geometrical quantity in space can be written as a functional of E_i^a . One of the simplest is the area $A_S[E_i^a]$ of a surface $S \subset \Sigma$ whose expression we derive in what follows. Given a two dimensional surface in $S \subset \Sigma$ —with normal

$$n_a = \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2} \epsilon_{abc} \quad (49)$$

where σ^1 and σ^2 are local coordinates on S —its area is given by

$$A_S[q^{ab}] = \int_S \sqrt{h} d\sigma^1 d\sigma^2, \quad (50)$$

where $h = \det(h_{ab})$ is the determinant of the metric $h_{ab} = q_{ab} - n^{-2}n_a n_b$ induced on S by q^{ab} . From equation (12) it follows that $\det(q^{ab}) = \det(E_i^a)$. Let us contract (12) with $n_a n_b$, namely

$$qq^{ab}n_a n_b = E_i^a E_j^b \delta^{ij} n_a n_b. \quad (51)$$

Now observe that $q^{nn} = q^{ab}n_a n_b$ is the nn -matrix element of the inverse of q_{ab} . Through the well known formula for components of the inverse matrix we have that

$$q^{nn} = \frac{\det(q_{ab} - n^{-2}n_a n_b)}{\det(q_{ab})} = \frac{h}{q}. \quad (52)$$

But $q_{ab} - n^{-2}n_a n_b$ is precisely the induced metric h_{ab} . Replacing q^{nn} back into (51) we conclude that

$$h = E_i^a E_j^b \delta^{ij} n_a n_b. \quad (53)$$

Finally we can write the area of S as an explicit functional of E_i^a :

$$A_S[E_i^a] = \int_S \sqrt{E_i^a E_j^b \delta^{ij} n_a n_b} d\sigma^1 d\sigma^2. \quad (54)$$

This simple expression for the area of a surface will be very important in the quantum theory.

3. The Dirac program: the non perturbative quantization of GR

The Dirac program^{38,29} applied to the quantization of generally covariant systems consists of the following steps^k:

- (i) Find a representation of the phase space variables of the theory as operators in an auxiliary or kinematical Hilbert space \mathcal{H}_{kin} satisfying the standard commutation relations, i.e., $\{ , \} \rightarrow -i/\hbar[,]$.
- (ii) Promote the constraints to (self-adjoint) operators in \mathcal{H}_{kin} . In the case of gravity we must quantize the seven constraints $G_i(A, E)$, $V_a(A, E)$, and $S(A, E)$.
- (iii) Characterize the space of solutions of the constraints and define the corresponding inner product that defines a notion of physical probability. This defines the so-called physical Hilbert space \mathcal{H}_{phys} .
- (iv) Find a (complete) set of gauge invariant observables, i.e., operators commuting with the constraints. They represent the questions that can be addressed in the generally covariant quantum theory.

3.1. A simple example: the reparametrized particle

Before going into the details of the definition of LQG we will present the general ideas behind the quantization of generally covariant systems using the simplest possible example: a non relativistic particle. A non relativistic particle can be treated

^kThere is another way to canonically quantize a theory with constraints that is also developed by Dirac. In this other formulation one solves the constraints at the classical level to identify the physical or reduced phase space to finally quantize the theory by finding a representation of the algebra of physical observables in the physical Hilbert space \mathcal{H}_{phys} . In the case of four dimensional gravity this alternative seem intractable due to the difficulty in identifying the true degrees of freedom of general relativity.

in a generally covariant manner by introducing a non physical parameter t and promoting the physical time T to a canonical variable. For a particle in one dimension the standard action

$$S(p_X, X) = \int dT \left[p_X \frac{dX}{dT} - H(p_X, X) \right] \quad (55)$$

can be replaced by the reparametrization invariant action

$$S_{rep}(P_T, P_X, T, X, N) = \int dt \left[p_T \dot{T} + p_X \dot{X} - N(p_T + H(p_X, X)) \right], \quad (56)$$

where the dot denotes derivative with respect to t . The previous action is invariant under the redefinition $t \rightarrow t' = f(t)$. The variable N is a Lagrange multiplier imposing the scalar constraint

$$C = p_T + H(p_X, X) = 0. \quad (57)$$

Notice the formal similarity with the action of general relativity (8) and (16). As in general relativity the Hamiltonian of the system $H_{rep} = N(p_T + H(p_X, X))$ is zero on shell. It is easy to see that on the constraint surface defined by (57) S_{rep} reduces to the standard S , and thus the new action leads to the same classical solutions. The constraint C is a generating function of infinitesimal t -reparametrizations (analog to diffeomorphisms in GR). This system is the simplest example of generally covariant system.

Let us proceed and analyze the quantization of this action according to the rules above.

- (i) We first define an auxiliary or *kinematical* Hilbert space \mathcal{H}_{kin} . In this case we can simply take $\mathcal{H}_{kin} = \mathcal{L}^2(\mathbb{R}^2)$. Explicitly we use (kinematic) wave functions of $\psi(X, T)$ and define the inner product

$$\langle \phi, \psi \rangle = \int dX dT \overline{\phi(X, T)} \psi(X, T). \quad (58)$$

We next promote the phase space variables to self adjoint operators satisfying the appropriate commutation relations. In this case the standard choice is that \hat{X} and \hat{T} act simply by multiplications and $\hat{p}_X = -i\hbar\partial/\partial X$ and $\hat{p}_T = -i\hbar\partial/\partial T$

- (ii) The constraint becomes—this step is highly non trivial in a field theory due to regularization issues:

$$\hat{C} = -i\hbar \frac{\partial}{\partial T} - \hbar^2 \frac{\partial^2}{\partial X^2} + V(X). \quad (59)$$

Notice that the constraint equation $\hat{C}|\psi\rangle = 0$ is nothing else than the familiar Schroedinger equation.

- (iii) The solutions of the quantum constraint are in this case the solutions of Schroedinger equation. As it is evident from the general form of Schroedinger equation we can characterize the set of solutions by specifying the initial wave

form at some $T = T_0$, $\psi(X) = \psi(X, T = T_0)$. The physical Hilbert space is therefore the standard $\mathcal{H}_{phys} = \mathcal{L}^2(\mathbb{R})$ with physical inner product

$$\langle \phi, \psi \rangle_p = \int dX \overline{\phi(X)} \psi(X). \quad (60)$$

The solutions of Schroedinger equation are not normalizable in \mathcal{H}_{kin} (they are not square-integrable with respect to (58) due to the time dependence imposed by the Schroedinger equation). This is a generic property of the solutions of the constraint when the constraint has continuous spectrum (think of the eigenstates of \hat{P} for instance).

- (iv) Observables in this setting are easy to find. We are looking for phase space functions commuting with the constraint. For simplicity assume for the moment that we are dealing with a free particle, i.e., $C = p_T + p_X^2/(2m)$. We have in this case the following two independent observables:

$$\hat{O}_1 = \hat{X} - \frac{\hat{p}_X}{m}(\hat{T} - T_0) \quad \text{and} \quad \hat{O}_2 = \hat{p}_X, \quad (61)$$

where T_0 is just a c -number. These are just the values of X and P at $T = T_0$. In the general case where $V(X) \neq 0$ the explicit form of these observables as functions of the phase space variables will depend on the specific interaction. Notice that in \mathcal{H}_{phys} , as defined above, the observables reduce to position $O_1 = X$ and momentum $O_2 = p_X$ as in standard quantum mechanics.

We have just reproduced standard quantum mechanics by quantizing the reparametrization invariant formulation. As advertised in the general framework of the Dirac program applied to generally covariant systems, the full dynamics is contained in the quantum constraints—here the Schroedinger equation.

3.2. *The program of loop quantum gravity*

A formal description of the implementation of Dirac's program in the case of gravity is presented in what follows. In the next section we will start a more detailed review of each of these steps.

- (i) In order to define the kinematical Hilbert space of general relativity in terms of the new variables we will choose the polarization where the connection is regarded as the configuration variable. The kinematical Hilbert space consists of a suitable set of functionals of the connection $\psi[A]$ which are square integrable with respect to a suitable (gauge invariant and diffeomorphism invariant) measure $d\mu_{AL}[A]$ (called Ashtekar-Lewandowski measure⁴⁰). The kinematical inner product is given by

$$\langle \psi, \phi \rangle = \mu_{AL}[\overline{\psi}\phi] = \int d\mu_{AL}[A] \overline{\psi}[A] \phi[A]. \quad (62)$$

In the next section we give the precise definition of \mathcal{H}_{kin} .

- (ii) Both the Gauss constraint and the diffeomorphism constraint have a natural (unitary) action on states on \mathcal{H}_{kin} . For that reason the quantization (and subsequent solution) is rather straightforward. The simplicity of these six-out-of-seven constraints is a special merit of the use of connection variables as will become transparent in the sequel.

The scalar constraint (25) does not have a simple geometric interpretation. In addition it is highly non linear which anticipates the standard UV problems that plague quantum field theory in the definition of products of fields (operator valued distributions) at a same point. Nevertheless, well defined versions of the scalar constraint have been constructed. The fact that these rigorously defined (free of infinities) operators exist is again intimately related to the kind of variables used for the quantization and some other special miracles occurring due to the background independent nature of the approach. We emphasize that the theory is free of divergences.

- (iii) *Quantum Einstein's equations* can be formally expressed now as:

$$\begin{aligned}\widehat{G}_i(A, E)|\Psi > &:= \widehat{D}_a E_i^a |\Psi > = 0 \\ \widehat{V}_a(A, E)|\Psi > &:= E_i^a \widehat{F}_{ab}^i(A) |\Psi > = 0, \\ \widehat{S}(A, E)|\Psi > &:= [\sqrt{\det E}^{-1} E_i^a E_j^b \widehat{F}_{ab}^{ij}(A) + \dots] |\Psi > = 0.\end{aligned}\tag{63}$$

As mentioned above, the space of solutions of the first six equations is well understood. The space of solutions of quantum scalar constraint remains an open issue in LQG. For some mathematically consistent definitions of \widehat{S} the characterization of the solutions is well understood¹⁹. The definition of the physical inner product is still an open issue. We will introduce the spin foam approach in Section 5 as a device for extracting solutions of the constraints producing at the same time a definition of the physical inner product in LQG. The spin foam approach also aims at the resolution of some difficulties appearing in the quantization of the scalar constraint that will be discussed in Section 4.3.2. It is yet not clear, however, whether these consistent theories reproduce general relativity in the semi-classical limit.

- (iv) Already in classical gravity the construction of gauge independent quantities is a subtle issue. At the present stage of the approach physical observables are explicitly known only in some special cases. Understanding the set of physical observables is however intimately related with the problem of characterizing the solutions of the scalar constraint described before. We will illustrate this by discussing simple examples of quasi-local Dirac observable in Section 4.3.2. For a vast discussion about this issue we refer the reader to Rovelli's book⁴.

4. Loop quantum gravity

4.1. Definition of the kinematical Hilbert space

4.1.1. The choice of variables: Motivation

As mentioned in the first item of the program formally described in the last section, we need to define the vector space of functionals of the connection and a notion of inner product to provide it with a Hilbert space structure of \mathcal{H}_{kin} . As emphasized in Section 2.4, a natural quantity associated with a connection consists of the holonomy along a path (48). We now give a more precise definition of it: Given a one dimensional oriented path $e : [0, 1] \subset \mathbb{R} \rightarrow \Sigma$ sending the parameter $s \in [0, 1] \rightarrow x^\mu(s)$, the holonomy $h_e[A] \in SU(2)$ is denoted

$$h_e[A] = P \exp - \int_e A, \quad (64)$$

where P denotes a path-ordered-exponential. More precisely, given the unique solution $h_e[A, s]$ of the ordinary differential equation

$$\frac{d}{ds} h_e[A, s] + \dot{x}^\mu(s) A_\mu h_e[A, s] = 0 \quad (65)$$

with the boundary condition $h_e[A, 0] = \mathbb{1}$, the holonomy along the path e is defined as

$$h_e[A] = h_e[A, 1]. \quad (66)$$

The previous differential equation has the form of a time dependent Schroedinger equation, thus its solution can be formally written in terms of the familiar series expansion

$$h_e[A] = \sum_{n=0}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \dot{x}^{\mu_1}(s_1) \cdots \dot{x}^{\mu_n}(s_n) A_{\mu_1}(s_1) \cdots A_{\mu_n}(s_n), \quad (67)$$

which is what the path ordered exponential denotes in (64). Let us list some important properties of the holonomy:

- (i) The definition of $h_e[A]$ is independent of the parametrization of the path e .
- (ii) The holonomy is a representation of the groupoid of oriented paths. Namely, the holonomy of a path given by a single point is the identity, given two oriented paths e_1 and e_2 such that the end point of e_1 coincides with the starting point of e_2 so that we can define $e = e_1 e_2$ in the standard fashion, then we have

$$h_e[A] = h_{e_1}[A] h_{e_2}[A], \quad (68)$$

where the multiplication on the right is the $SU(2)$ multiplication. We also have that

$$h_{e^{-1}}[A] = h_e^{-1}[A]. \quad (69)$$

- (iii) The holonomy has a very simple behavior under gauge transformations. It is easy to check from (29) that under a gauge transformation generated by the Gauss constraint, the holonomy transforms as

$$h'_e[A] = g(x(0)) h_e[A] g^{-1}(x(1)). \quad (70)$$

- (iv) The holonomy transforms in a very simple way under the action of diffeomorphisms (transformations generated by the vector constraint (24)). Given $\phi \in \text{Diff}(\Sigma)$ we have

$$h_e[\phi^* A] = h_{\phi^{-1}(e)}[A], \quad (71)$$

where $\phi^* A$ denotes the action of ϕ on the connection. In other words, transforming the connection with a diffeomorphism is equivalent to simply ‘moving’ the path with ϕ^{-1} .

Geometrically the holonomy $h_e[A]$ is a functional of the connection that provides a rule for the parallel transport of $SU(2)$ spinors along the path e . If we think of it as a functional of the path e it is clear that it captures all the information of the field A_a^i . In addition it has very simple behavior under the transformations generated by six of the constraints (24),(25), and (26). For these reasons the holonomy is a natural choice of basic functional of the connection.

4.1.2. The algebra of basic (kinematic) observables

Shifting the emphasis from connections to holonomies leads to the concept of *generalized connections*. A generalized connection is an assignment of $h_e \in SU(2)$ to any path $e \subset \Sigma$. In other words the fundamental observable is taken to be the holonomy itself and not its relationship (64) to a smooth connection. The algebra of kinematical observables is defined to be the algebra of the so-called *cylindrical functions* of generalized connections denoted Cyl . The latter algebra can be written as the union of the set of functions of generalized connections defined on graphs $\gamma \subset \Sigma$, namely

$$\text{Cyl} = \cup_{\gamma} \text{Cyl}_{\gamma}, \quad (72)$$

where Cyl_{γ} is defined as follows.

A graph γ is defined as a collection of paths $e \subset \Sigma$ (e stands for *edge*) meeting at most at their endpoints. Given a graph $\gamma \subset \Sigma$ we denote by N_e the number of paths or edges that it contains. An element $\psi_{\gamma,f} \in \text{Cyl}_{\gamma}$ is labelled by a graph γ and a smooth function $f : SU(2)^{N_e} \rightarrow \mathbb{C}$, and it is given by a functional of the connection defined as

$$\psi_{\gamma,f}[A] := f(h_{e_1}[A], h_{e_2}[A], \dots, h_{e_{N_e}}[A]), \quad (73)$$

where e_i for $i = 1, \dots, N_e$ are the edges of the corresponding graph γ . The symbol \cup_{γ} in (72) denotes the union of Cyl_{γ} for all graphs in Σ . This is the algebra of basic

observables upon which we will base the definition of the kinematical Hilbert space \mathcal{H}_{kin} .

Before going into the construction of the representation of Cyl that defines \mathcal{H}_{kin} , it might be useful to give a few examples of cylindrical functions. For obvious reason, $SU(2)$ gauge invariant functions of the connection will be of particular interest in the sequel. The simplest of such functions is the Wilson loop: given a closed loop γ the Wilson loop is given by the trace of the holonomy around the loop, namely

$$W_\gamma[A] := \text{Tr}[h_\gamma[A]]. \quad (74)$$

Equation (70) and the invariance of the trace implies the $W_\gamma[A]$ is gauge invariant. The Wilson loop $W_\gamma[A]$ is an element of $\text{Cyl}_\gamma \subset \text{Cyl}$ according to the previous definition. The graph consists of a single closed edge ($e = \gamma$) and an example is shown in Figure 2. Notice, however, that we can also define $W_\gamma[A]$ as

$$W_\gamma[A] := \text{Tr}[h_{e_1}[A]h_{e_2}[A]], \quad (75)$$

using (68) in which case $W_\gamma[A] \in \text{Cyl}_{\gamma'}$ (γ' is illustrated in the center of Figure 2). Moreover, we can also think of $W_\gamma[A] \in \text{Cyl}_{\gamma''}$ (γ'' is shown on the right of Figure 2) as a function of $h_{e_1}[A]$ and $h_{e_2}[A]$ and $h_{e_3}[A]$ (with trivial dependence on the third argument). There are many ways to represent an element of Cyl as a cylindrical function on a graph. This flexibility in choosing the graph will be important in the definition of \mathcal{H}_{kin} and its inner product in the following section.

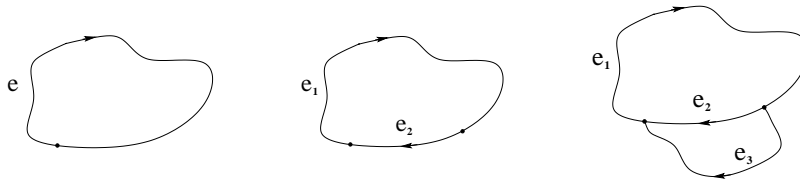


Fig. 2. An example of three different graphs on which the Wilson loop function (74) can be defined. The distinction must be physically irrelevant.

Let us come back to our examples. There is a simple generalization of the previous gauge invariant function. Given an arbitrary representation matrix M of $SU(2)$, then clearly $W_\gamma^M[A] = \text{Tr}[M(h_e[A])]$ is a gauge invariant cylindrical function. Unitary irreducible representation matrices of spin j will be denoted by $\overset{j}{\Pi}_{mm'}$ for $-j \leq m, m' \leq j$. The cylindrical function

$$W_\gamma^j[A] := \text{Tr}[\overset{j}{\Pi}(h_e[A])] \quad (76)$$

is the simplest example of *spin network* function^{41,42,43,44}. This function is represented as on the left of Figure 3.

A more sophisticated spin network function can be associated to the graph on the right of Figure 2. We take different representation matrices of spins 1, 1/2 and

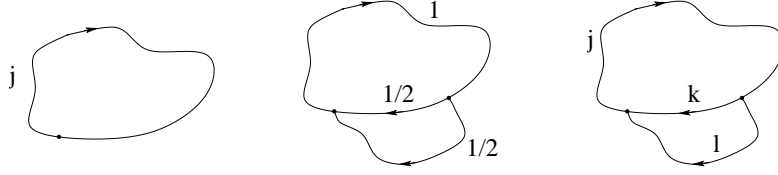


Fig. 3. Examples of spin networks.

1/2 evaluate them on the holonomy along e_1 , e_2 , and e_3 respectively. We define

$$\Theta_{e_1 \cup e_2 \cup e_3}^{1,1/2,1/2}[A] = \overset{1}{\prod} (h_{e_1}[A])^{ij} \overset{1/2}{\prod} (h_{e_2}[A])_{AB} \overset{1/2}{\prod} (h_{e_3}[A])_{CD} \sigma_i^{AC} \sigma_j^{BD}, \quad (77)$$

where $i, j = 1, 2, 3$ are vector indices, $A, B, C, D = 1, 2$ are spinor indices, sum over repeated indices is understood and σ_i^{AC} are Pauli matrices. It is easy to check that $\Theta_{e_1 \cup e_2 \cup e_3}^{1,1/2,1/2}[A]$ is gauge invariant. This is because the Pauli matrices are invariant tensors in the tensor product of representations $1 \otimes 1/2 \otimes 1/2$ which is where gauge transformations act on the nodes of the graph $e_1 \cup e_2 \cup e_3$. Such spin network function is illustrated on the middle of Figure 3. We can generalize this to arbitrary representations. Given an invariant tensor $\iota \in j \otimes k \otimes l$ the cylindrical function

$$\Theta_{e_1 \cup e_2 \cup e_3}^{j,k,l}[A] = \overset{j}{\prod} (h_{e_1}[A])_{m_1 n_1} \overset{k}{\prod} (h_{e_2}[A])_{m_2 n_2} \overset{l}{\prod} (h_{e_3}[A])_{m_3 n_3} \iota^{m_1 m_2 m_3 n_1 n_2 n_3} \quad (78)$$

is gauge invariant by construction. The example is shown on the right of Figure 3.

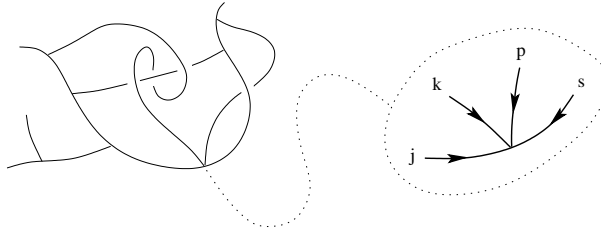


Fig. 4. Schematic representation of the construction of a spin network. To each node we associate an invariant vector in the tensor product of irreducible representations converging at the node. In this case we take $\iota^{n_1 n_2 n_3 n_4} \in j \otimes k \otimes p \otimes s$, and the relevant piece of spin network function is $\overset{j}{\prod} (h_{e_1}[A])_{m_1 n_1} \overset{k}{\prod} (h_{e_2}[A])_{m_2 n_2} \overset{p}{\prod} (h_{e_3}[A])_{m_3 n_3} \overset{s}{\prod} (h_{e_4}[A])_{m_4 n_4} \iota^{n_1 n_2 n_3 n_4}$.

One can generalize the construction of these examples to the definition of *spin networks* on arbitrary graphs $\gamma \subset \Sigma$. The general construction is analogous to the one in the previous examples. One labels the set of edges $e \subset \gamma$ with spins $\{j_e\}$. To each node $n \subset \gamma$ one assigns an invariant tensor, also called an *intertwiner*, ι_n in the tensor product of representations labelling the edges converging at the

corresponding node (see Figure 4). The spin network function is defined

$$s_{\gamma, \{j_e\}, \{\iota_n\}}[A] = \bigotimes_{n \in C\gamma} \iota_n \bigotimes_{e \in C\gamma} \Pi^{j_e} (h_e[A]), \quad (79)$$

where the indices of representation matrices and invariant tensors is left implicit in order to simplify the notation. An example is shown in Figures 4 and 5.

Intertwiners in the tensor product of an arbitrary number of irreducible representations can be expressed in terms of basic intertwiners between three irreducible representations. In the case of $SU(2)$ the latter are uniquely defined up to a normalization; they are simply related to Clebsh-Gordon coefficients— $\text{Inv}[j_1 \otimes j_2 \otimes j_3]$ is either trivial or one dimensional according to the standard rules for addition of angular momentum. The construction is illustrated on the left of Figure 5, on the right we show an explicit example of spin network with the nodes decomposed in terms of three valent intertwiners..

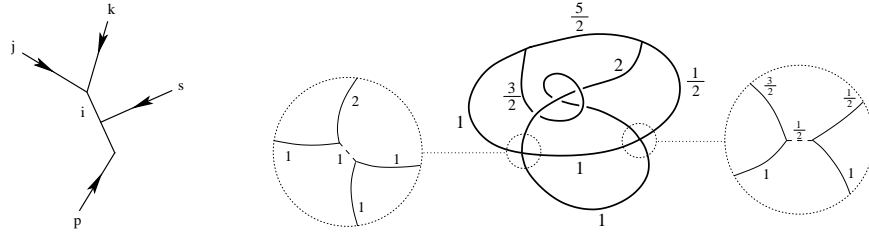


Fig. 5. On the left: any invariant vector can be decomposed in terms of the (unique up to normalization) three valent ones. At each three node the standard rules of addition of angular momentum must be satisfied for a non vanishing intertwiner to exist. On the right: an example of spin network with the explicit decomposition of intertwiners.

So far we have introduced the algebra of functionals of (generalized) connections Cyl . Spin networks were presented here as special examples of elements of Cyl , which in addition are $SU(2)$ gauge invariant. In the Subsection 4.1.5 we will show how spin network functions define a complete basis of \mathcal{H}_{kin} . But in order to do that we must define \mathcal{H}_{kin} .

4.1.3. *The Ashtekar-Lewandowski representation of Cyl*

The cylindrical functions introduced in the previous section are the candidates for states in \mathcal{H}_{kin} . In this section we define \mathcal{H}_{kin} and provide a representation of the algebra Cyl . Essentially what we need is the notion of a measure in the space of generalized connections in order to give a meaning to the formal expression (62) and thus obtain a definition of the kinematical inner product. In order to do that we introduce a positive normalized state (state in the algebraic QFT sense) μ_{AL} on the (C^* -algebra) Cyl as follows. Given a cylindrical function $\psi_{\gamma, f}[A] \in \text{Cyl}$ (as in

(73) $\mu_{AL}(\psi_{\gamma,f})$ is defined as

$$\mu_{AL}(\psi_{\gamma,f}) = \int \prod_{e \in \gamma} dh_e f(h_{e_1}, h_{e_2}, \dots, h_{e_{N_e}}), \quad (80)$$

where $h_e \in SU(2)$ and dh is the (normalized) Haar measure of $SU(2)$ ¹. The measure μ_{AL} is clearly normalized as $\mu_{AL}(1) = 1$ and positive

$$\mu_{AL}(\overline{\psi_{\gamma,f}} \psi_{\gamma,f}) = \int \prod_{e \in \gamma} dh_e \overline{f(h_{e_1}, h_{e_2}, \dots, h_{e_{N_e}})} f(h_{e_1}, h_{e_2}, \dots, h_{e_{N_e}}) \geq 0. \quad (81)$$

Using the properties of μ_{AL} we introduce the inner product

$$\begin{aligned} \langle \psi_{\gamma,f}, \psi_{\gamma',g} \rangle &:= \mu_{AL}(\overline{\psi_{\gamma,f}} \psi_{\gamma',g}) = \\ &= \int \prod_{e \in \Gamma_{\gamma\gamma'}} dh_e \overline{f(h_{e_1}, \dots, h_{e_{N_e}})} g(h_{e_1}, \dots, h_{e_{N_e}}), \end{aligned} \quad (82)$$

where we use Dirac notation and the cylindrical functions become wave functionals of the connection corresponding to kinematical states $\psi_{\gamma',g}[A] = \langle A, \psi_{\gamma',g} \rangle = g(h_{e_1}, \dots, h_{e_{N_e}})$, and $\Gamma_{\gamma\gamma'}$ is any graph such that both $\gamma \subset \Gamma_{\gamma\gamma'}$ and $\gamma' \subset \Gamma_{\gamma\gamma'}$. The state μ_{AL} is called the Ashtekar-Lewandowski measure⁴⁰. The previous equation is the rigorous definition of (62). The measure μ_{AL} —through the GNS construction⁴⁵—gives a faithful representation of the algebra of cylindrical functions (i.e., (81) is zero if and only if $\psi_{\gamma,f}[A] = 0$). The kinematical Hilbert space \mathcal{H}_{kin} is the Cauchy completion of the space of cylindrical functions Cyl in the Ashtekar-Lewandowski measure. In other words, in addition to cylindrical functions we add to \mathcal{H}_{kin} the limits of all the Cauchy convergent sequences in the μ_{AL} norm. The operators depending only on the connection act simply by multiplication in the Ashtekar-Lewandowski representation. This completes the definition of the kinematical Hilbert space \mathcal{H}_{kin} .

4.1.4. An orthonormal basis of \mathcal{H}_{kin} .

In this section we would like to introduce a very simple basis of \mathcal{H}_{kin} as a preliminary step in the construction of a basis of the Hilbert space of solutions of the Gauss constraint \mathcal{H}_{kin}^g . At this stage this is a simple consequence of the Peter-Weyl theorem⁴⁶. The Peter-Weyl theorem can be viewed as a generalization of Fourier theorem for functions on S^1 . It states that, given a function $f \in \mathcal{L}^2[SU(2)]$, it can

¹The Haar measure of $SU(2)$ is defined by the following properties:

$$\int_{SU(2)} dg = 1, \quad \text{and} \quad dg = d(\alpha g) = d(g\alpha) = dg^{-1} \quad \forall \alpha \in SU(2).$$

be expressed as a sum over unitary irreducible representations of $SU(2)$, namely ^m

$$f(g) = \sum_j \sqrt{2j+1} f_j^{mm'} \overset{j}{\Pi}_{mm'}(g), \quad (83)$$

where

$$f_j^{mm'} = \sqrt{2j+1} \int_{SU(2)} dg \overset{j}{\Pi}_{m'm}(g^{-1}) f(g), \quad (84)$$

and dg is the Haar measure of $SU(2)$. This defines the harmonic analysis on $SU(2)$. The completeness relation

$$\delta(gh^{-1}) = \sum_j (2j+1) \overset{j}{\Pi}_{mm'}(g) \overset{j}{\Pi}_{m'm}(h^{-1}) = \sum_j (2j+1) \text{Tr}[\overset{j}{\Pi}(gh^{-1})], \quad (85)$$

follows. The previous equations imply the orthogonality relation for unitary representations of $SU(2)$

$$\int_{SU(2)} dg \phi_{m'm}^j \phi_{q'q}^{j'} = \delta_{jj'} \delta_{mq} \delta_{m'q'}, \quad (86)$$

where we have introduced the normalized representation matrices $\phi_{mn}^j := \sqrt{2j+1} \overset{j}{\Pi}_{mn}$ for convenience. Given an arbitrary cylindrical function $\psi_{\gamma,f}[A] \in \text{Cyl}$ we can use the Peter-Weyl theorem and write

$$\begin{aligned} \psi_{\gamma,f}[A] &= f(h_{e_1}[A], h_{e_2}[A], \dots, h_{e_{N_e}}[A]) = \\ &= \sum_{j_1 \dots j_{N_e}} \int_{j_1 \dots j_{N_e}}^{m_1 \dots m_{N_e}, n_1 \dots n_{N_e}} \phi_{m_1 n_1}^{j_1}(h_{e_1}[A]) \dots \phi_{m_{N_e} n_{N_e}}^{j_{N_e}}(h_{e_{N_e}}[A]), \end{aligned} \quad (87)$$

where according to (84) $\int_{j_1 \dots j_{N_e}}^{m_1 \dots m_{N_e}, n_1 \dots n_{N_e}}$ is just given by the kinematical inner product of the cylindrical function with the tensor product of irreducible representations, namely (82)

$$\int_{j_1 \dots j_{N_e}}^{m_1 \dots m_{N_e}, n_1 \dots n_{N_e}} = \langle \phi_{m_1 n_1}^{j_1} \dots \phi_{m_{N_e} n_{N_e}}^{j_{N_e}}, \psi_{\gamma,f} \rangle, \quad (88)$$

where \langle, \rangle is the kinematical inner product introduced in (82). We have thus proved that the product of components of (normalized) irreducible representations $\prod_{i=1}^{N_e} \phi_{m_i n_i}^{j_i}[h_{e_i}]$ associated with the N_e edges $e \subset \gamma$ (for all values of the spins j and $-j \leq m, n \leq j$ and for any graph γ) is a complete orthonormal basis of \mathcal{H}_{kin} !

^mThe link with the $U(1)$ case is direct: for $f \in \mathcal{L}^2[U(1)]$ we have $f(\theta) = \sum_n f_n \exp(in\theta)$, where $\exp(in\theta)$ are unitary irreducible representations of $U(1)$ and $f_n = (2\pi)^{-1} \int d\theta \exp(-in\theta) f(\theta)$. The measure $(2\pi)^{-1} d\theta$ is the Haar measure of $U(1)$.

4.1.5. *Solutions of the Gauss constraint: \mathcal{H}_{kin}^g and spin network states*

We are now interested in the solutions of the quantum Gauss constraint; the first three of quantum Einstein's equations. These solutions are characterized by the states in \mathcal{H}_{kin} that are $SU(2)$ gauge invariant. These solutions define a new Hilbert space that we call \mathcal{H}_{kin}^g . We leave the subindex *kin* to keep in mind that there are still constraints to be solved on the way to \mathcal{H}_{phys} . In previous sections we already introduced spin network states as natural $SU(2)$ gauge invariant functionals of the connection^{42,43,44,47}. Now we will show how these are in fact a complete set of orthogonal solutions of the Gauss constraint, i.e., a basis of \mathcal{H}_{kin}^g .

The action of the Gauss constraint is easily represented in \mathcal{H}_{kin} . At this stage it is simpler to represent finite $SU(2)$ transformations on elements of \mathcal{H}_{kin} (from which the infinitesimal ones can be easily inferred) using (70). Denoting $\mathcal{U}_G[g]$ the operator generating a local $g(x) \in SU(2)$ transformation then its action can be defined directly on the elements of the basis of \mathcal{H}_{kin} defined above, thus

$$\mathcal{U}_G[g]\phi_{mn}^j[h_e] = \phi_{mn}^j[g_s h_e g_t^{-1}], \quad (89)$$

where g_s is the value of $g(x)$ at the source point of the edge e and g_t the value of $g(x)$ at the target. From the previous equation one can infer the action on an arbitrary basis element, namely

$$\mathcal{U}_G[g] \prod_{i=1}^{N_e} \phi_{m_i n_i}^{j_i}[h_{e_i}] = \prod_{i=1}^{N_e} \phi_{m_i n_i}^{j_i}[g_{s_i} h_{e_i} g_{t_i}^{-1}]. \quad (90)$$

Now by definition of the scalar product (82) and due to the invariance of the Haar measure (see Footnote m) the reader can easily prove that $\mathcal{U}_G[g]$ is a unitary operator. From the definition it also follows that

$$\mathcal{U}_G[g_2]\mathcal{U}_G[g_1] = \mathcal{U}_G[g_1 g_2]. \quad (91)$$

The projection operator onto the set of states that are solutions of the Gauss constraint can be obtained by group averaging techniques. We can denote the projector P_G by

$$P_G = \int D[g] \mathcal{U}_G[g], \quad (92)$$

where the previous expression denotes a formal integration over all $SU(2)$ transformations. Its rigorous definition is given by its action on elements of Cyl . From equation (89) the operator $\mathcal{U}_G[g]$ acts on $\psi_{\gamma,g} \in \text{Cyl}$ at the end points of the edges $e \subset \gamma$, and therefore, so does P_G . The action of P_G on a given (cylindrical) state $\psi_{\gamma,f} \in \mathcal{H}_{kin}$ can therefore be factorized as follows:

$$P_G \psi_{\gamma,f} = \prod_{n \subset \gamma} P_G^n \psi_{\gamma,f}, \quad (93)$$

where P_G^n acts non trivially only at the node $n \subset \gamma$. In this way we can define the action of P_G by focusing our attention to a single node $n \subset \gamma$. For concreteness let

us concentrate on the action of $P_{\mathcal{G}}$ on an element of $\psi_{\gamma,f} \in \mathcal{H}_{kin}$ defined on the graph illustrated in Figure 4. The state $\psi_{\gamma,f} \in \mathcal{H}_{kin}$ admits an expansion in terms of the basis states as in (87). In particular we concentrate on the action of $P_{\mathcal{G}}$ at the four valent node thereto emphasized, let's call it $n_0 \subset \gamma$. In order to do that we can factor out of (87) the (normalized) representation components ϕ_{mn}^j corresponding to that particular node and write

$$\begin{aligned} \psi_{\gamma,f}[A] &= \\ &= \sum_{j_1 \dots j_4} (\phi_{m_1 n_1}^{j_1}(h_{e_1}[A]) \dots \phi_{m_4 n_4}^{j_4}(h_{e_4}[A])) \times [\text{REST}]_{j_1 \dots j_4}^{m_1 \dots m_4, n_1 \dots n_4}[A], \end{aligned} \quad (94)$$

where $[\text{REST}]_{j_1 \dots j_4}^{m_1 \dots m_4, n_1 \dots n_4}[A]$ denotes what is left in this factorization, and e_1 to e_4 are the four edges converging at n_0 (see Figure 4). We can define the meaning of (92) by giving the action of $P_{\mathcal{G}}^{n_0}$ on $\phi_{m_1 n_1}^{j_1}(h_{e_1}[A]) \dots \phi_{m_4 n_4}^{j_4}(h_{e_4}[A])$ as the action on a general state can be naturally extended from there using (93). Thus we define

$$P_{\mathcal{G}}^{n_0} \phi_{m_1 n_1}^{j_1}(h_{e_1}[A]) \dots \phi_{m_4 n_4}^{j_4}(h_{e_4}[A]) = \int dg \phi_{m_1 n_1}^{j_1}(gh_{e_1}[A]) \dots \phi_{m_4 n_4}^{j_4}(gh_{e_4}[A]), \quad (95)$$

where dg is the Haar measure of $SU(2)$. Using the fact that

$$\phi_{mn}^j(gh[A]) = \prod_{mq}^j(g) \phi_{qn}^j(h[A]) \quad (96)$$

the action of $P_{\mathcal{G}}^{n_0}$ can be written as

$$\begin{aligned} P_{\mathcal{G}}^{n_0} \phi_{m_1 n_1}^{j_1}(h_{e_1}[A]) \dots \phi_{m_4 n_4}^{j_4}(h_{e_4}[A]) &= \\ = P_{m_1 \dots m_4, q_1 \dots q_4}^{n_0} \phi_{q_1 n_1}^{j_1}(h_{e_1}[A]) \dots \phi_{q_4 n_4}^{j_4}(h_{e_4}[A]), \end{aligned} \quad (97)$$

where

$$P_{m_1 \dots m_4, q_1 \dots q_4}^{n_0} = \int dg \prod_{m_1 q_1}^{j_1}(g) \dots \prod_{m_4 q_4}^{j_4}(g). \quad (98)$$

If we denote $V_{j_1 \dots j_4}$ the vector space where the representation $j_1 \otimes \dots \otimes j_4$ act, then previous equation defines a map $P^{n_0} : V_{j_1 \dots j_4} \rightarrow V_{j_1 \dots j_4}$. Using the properties of the Haar measure given in Footnote m one can show that the map P^{n_0} is indeed a projection (i.e., $P^{n_0} P^{n_0} = P^{n_0}$). Moreover, we also have

$$\begin{aligned} P_{m_1 \dots m_4, q_1 \dots q_4}^{n_0} \prod_{q_1 n_1}^{j_1}(g) \dots \prod_{q_4 n_4}^{j_4}(g) &= \\ = \prod_{m_1 q_1}^{j_1}(g) \dots \prod_{m_4 q_4}^{j_4}(g) P_{q_1 \dots q_4, n_1 \dots n_4}^{n_0} = P_{m_1 \dots m_4, n_1 \dots n_4}^{n_0}, \end{aligned} \quad (99)$$

i.e. P^{n_0} is right and left invariant. This implies that $P^{n_0} : V_{j_1 \dots j_4} \rightarrow \text{Inv}[V_{j_1 \dots j_4}]$, i.e., the projection from $V_{j_1 \dots j_4}$ onto the $(SU(2))$ invariant component of the finite dimensional vector space. We can choose an orthogonal set of invariant vectors $\iota_{m_1 \dots m_4}^\alpha$ (where α labels the elements), in other words an orthonormal basis for $\text{Inv}[V_{j_1 \dots j_4}]$ and write

$$P_{m_1 \dots m_4, n_1 \dots n_4}^{n_0} = \sum_{\alpha} \iota_{m_1 \dots m_4}^\alpha \iota_{m_1 \dots m_4}^{\alpha*}, \quad (100)$$

where $*$ denotes the dual basis element. In the same way the action P_G on a node $n \subset \gamma$ of arbitrary valence κ is governed by the corresponding P^n given generally by

$$P_{m_1 \dots m_\kappa, n_1 \dots n_\kappa}^n = \sum_{\alpha_\kappa} \iota_{m_1 \dots m_\kappa}^{\alpha_\kappa} \iota_{m_1 \dots m_\kappa}^{\alpha_\kappa *}. \quad (101)$$

According to the tree decomposition of intertwiners in terms of three valent invariant vectors described around Figure 5, α_κ is a $(\kappa - 3)$ -uple of spins.

Any solution of the Gauss constraint can be written as $P_G \psi$ for $\psi \in \mathcal{H}_{kin}$. Equation (93) plus the obvious generalization of (97) for arbitrary nodes implies that the result of the action of P_G on elements of \mathcal{H}_{kin} can be written as a linear combination of products of representation matrices ϕ_{mn}^j contracted with intertwiners, i.e. spin network states as introduced as examples of elements of Cyl in (79). Spin network states therefore form a complete basis of the Hilbert space of solutions of the quantum Gauss law $\mathcal{H}_{kin}^\mathcal{G}$!

4.2. Geometric operators: quantization of the triad

We have introduced the set of basic configuration observables as the algebra of cylindrical functions of the generalized connections, and have defined the kinematical Hilbert space through the Ashtekar-Lewandowski representation. By considering finite gauge transformations we avoided quantizing the Gauss constraint in the previous section, avoiding for simplicity, and for the moment, the quantization of the triad field E_i^a present in (26). In this section we will quantize the triad field and will define a set of geometrical operators that lead to the main physical prediction of LQG: discreteness of geometry eigenvalues.

The triad E_i^a naturally induces a two form with values in the Lie algebra of $SU(2)$, namely, $E_i^a \epsilon_{abc}$. In the quantum theory E_i^a becomes an operator valued distribution. In other words we expect integrals of the triad field with suitable test functions to be well defined self adjoint operators in \mathcal{H}_{kin} . The two form naturally associated to E_i^a suggests that the smearing should be defined on two dimensional surfaces:

$$\widehat{E}[S, \alpha] = \int_S d\sigma^1 d\sigma^2 \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^2} \alpha^i \widehat{E}_i^a \epsilon_{abc} = -i\hbar\kappa\gamma \int_S d\sigma^1 d\sigma^2 \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^2} \alpha^i \frac{\delta}{\delta A_c^i} \epsilon_{abc}, \quad (102)$$

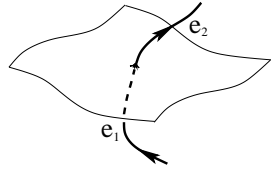
where α^i is a smearing function with values on the Lie algebra of $SU(2)$. The previous expression corresponds to the natural generalization of the notion of electric flux operator in electromagnetism. In order to study the action of (102) in \mathcal{H}_{kin} we notice that

$$\begin{aligned} \frac{\delta}{\delta A_c^i} h_e[A] &= \frac{\delta}{\delta A_c^i} \left(\text{P exp} \int ds \dot{x}^d(s) A_d^k \tau_k \right) = \\ &= \int ds \dot{x}^c(s) \delta^{(3)}(x(s) - x) h_{e_1}[A] \tau_i h_{e_2}[A], \end{aligned} \quad (103)$$

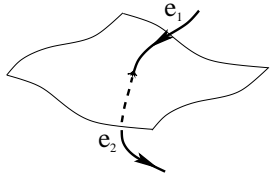
where $h_{e_1}[A]$ and $h_{e_2}[A]$ are the holonomy along the two new edges defined by the point at which the triad acts. Therefore

$$\begin{aligned} \widehat{E}[S, \alpha]h_e[A] &= \\ &= -i8\pi\ell_p^2\gamma \int d\sigma^1 d\sigma^2 d\sigma^3 \frac{\partial x^a}{\partial \sigma^1} \frac{\partial x^b}{\partial \sigma^2} \frac{\partial x^c}{\partial \sigma^3} \epsilon_{abc} \delta^{(3)}(x(\sigma), x(s)) \alpha^i h_{e_1}[A] \tau_i h_{e_2}[A]. \end{aligned} \quad (104)$$

Finally using the definition of the delta distribution we obtain a very simple expression for the action of the flux operator on the holonomy integrating the previous expression. In the cases of interest the result is:

$$\widehat{E}[S, \alpha]h_e[A] = -i8\pi\ell_p^2\gamma \alpha^i h_{e_1}[A] \tau_i h_{e_2}[A] \quad , \quad (105)$$


and

$$\widehat{E}[S, \alpha]h_e[A] = i8\pi\ell_p^2\gamma \alpha^i h_{e_1}[A] \tau_i h_{e_2}[A] \quad , \quad (106)$$


and $\widehat{E}[S, \alpha]h_{e_1}[A] = 0$ when e is tangential to S or $e \cap S = 0$.

From its action on the holonomy, and using basic $SU(2)$ representation theory, one can easily obtain the action of $\widehat{E}[S, \alpha]$ to spin network states and therefore to any state in \mathcal{H}_{kin} . Using (82) one can also verify that $\widehat{E}[S, \alpha]$ is self-adjoint. The reader can also verify that it is $SU(2)$ gauge covariant. The operators $\widehat{E}[S, \alpha]$ for all surfaces S and all smearing functions α contain all the information of the quantum Riemannian geometry of Σ . In terms of the operators $\widehat{E}[S, \alpha]$ we can construct any geometric operator.

4.2.1. Quantization of the Gauss constraint

We have already imposed the Gauss constraint in Section 4.1.5 by direct construction of the Hilbert space of $SU(2)$ gauge invariant states \mathcal{H}_{kin}^g with a natural complete basis given by the spin network states. Here we show that an important identity for flux operators follows from gauge invariance. Given a spin network node n where N_n edges converge, take a sphere S of radius ϵ (defined in some local coordinates) centered at the corresponding node (see Figure 6). The identity follows from the fact that for a gauge invariant node

$$\lim_{\epsilon \rightarrow 0} \widehat{E}_S(\alpha)|\psi\rangle = \sum_{i=1}^{N_n} \widehat{E}_{e_i}(\alpha)|\psi\rangle = 0, \quad (107)$$

where N_n is the number of edges at the node, and $E_{e_i}(\alpha)$ is the flux operator through a piece of the sphere that is punctured by only the edge e_i . If we partition the sphere

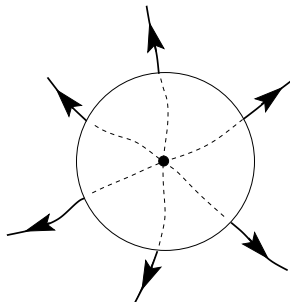


Fig. 6. The Gauss constraint imposes the net flux of non Abelian electric field be zero around nodes.

in pieces that are punctured by only one edge, using (105) at each edge one notices that (107) produces the first order term in an infinitesimal gauge transformation $g_\alpha = \mathbb{1} - \alpha^i \tau_i \in SU(2)$ at the corresponding node: the operator acting in (107) is indeed the quantum Gauss constraint action on the given node! Because the node is gauge invariant the action of the such operator vanishes identically. The total quantum flux of non-Abelian electric field must vanish according the Gauss constraint.

4.2.2. Quantization of the area

The simplest of the geometric operators is the area of a two-dimensional surface^{21,22,48} $S \subset \Sigma$ which classically depends on the triad E_i^a as in (54). If we introduce a decomposition of S in two-cells, we can write the integral defining the area as the limit of a Riemann sum, namely

$$A_S = \lim_{N \rightarrow \infty} A_S^N \quad (108)$$

where the Riemann sum can be expressed as

$$A_S^N = \sum_{I=1}^N \sqrt{E_i(S_I) E^i(S_I)} \quad (109)$$

where N is the number of cells, and $E_i(S_I)$ corresponds to the flux of E_i^a through the I -th cell. The reader is invited to check that the previous limit does in fact define the area of S in classical geometry. The previous expression for the area sets the path to the definition of the corresponding quantum operator as it is written in terms of the flux operators that we defined in the previous section. The quantum area operator then simply becomes

$$\widehat{A}_S = \lim_{N \rightarrow \infty} \widehat{A}_S^N, \quad (110)$$

where we simply replace the classical $E_i(S_I)$ by $\widehat{E}_i(S_I)$ according to (102). The important action to study is that of $\widehat{E}_i(S_I)\widehat{E}^i(S_I)$ which on the holonomy along a path that crosses S_I only once is

$$\widehat{E}_i(S_I)\widehat{E}^i(S_I)h_e[A] = i^2(8\pi\ell_p^2\gamma)^2 h_{e_1}[A]\tau_i\tau^i h_{e_2}[A] = (8\pi\ell_p^2\gamma)^2(3/4)h_e[A], \quad (111)$$

where we have used that $\tau_i = i\sigma_i/2$. The action of the square of the flux through the cell S_I is diagonal on such holonomy! Using the definition of the unitary irreducible representation of $SU(2)$ it follows that

$$\widehat{E}_i(S_I)\widehat{E}^i(S_I)\phi^j(h_e[A])_{mn} = (8\pi\ell_p^2\gamma)^2(j(j+1))\phi^j(h_e[A])_{mn}, \quad (112)$$

when the edge is that of an arbitrary spin network state. The remaining important case is when a spin network node is on S_I . A careful analysis shows that the action is still diagonal in this case⁴⁸. Notice that the cellular decomposition is chosen so that in the limit $N \rightarrow \infty$ each S_I is punctured at most at a single point by either an edge (the case studied here) or a node.

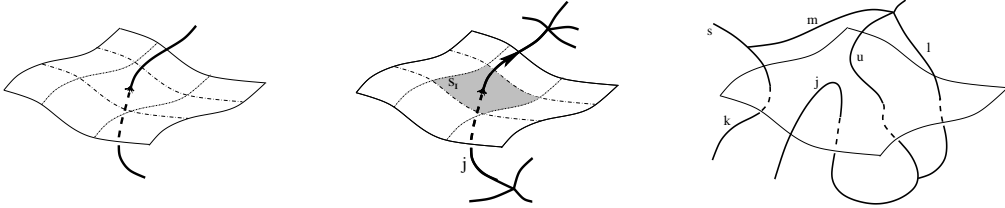


Fig. 7. On the left: the regularization of (110) is defined so that in the limit in which the two cells are shrunk they are punctured by at most one edge. Center: The simplest eigenstate of the area of S is illustrated, a two cell at finite regulator is emphasized. On the right: a generic eigenstate of the area.

It is then a straightforward exercise to show that the action of the area operator is diagonalized by the spin network states. Spin network states are the eigenstates of the quantum area operator! We have

$$\widehat{A}_S|\psi\rangle = 8\pi\ell_p^2\gamma\sqrt{j(j+1)}|\psi\rangle, \quad (113)$$

for a single puncture and more generally

$$\widehat{A}_S|\psi\rangle = 8\pi\ell_p^2\gamma\sum_p\sqrt{j_p(j_p+1)}|\psi\rangle. \quad (114)$$

The eigenvalues when nodes lay on S are also known in closed form. We do not analyze that case here for lack of space; however, the eigenvalues can be computed in a direct manner using the tools that have been given here. The reader is encouraged to try although the full answer can be found explicitly in the literature⁴⁸. Notice that the spectrum of the area operator depends on the value of the Immirzi parameter γ (introduced in (21)). This is a general property of geometric operators. The spectrum of the area operator is clearly discrete.

4.2.3. Quantization of the volume

The volume of a three dimensional region $B \subset \Sigma$ is classically given by

$$V_B = \int_B \sqrt{q} d^3x, \quad (115)$$

and it can be expressed in terms of the triad operator. In fact using (12) we conclude that

$$q = |\det(E)| = \left| \frac{1}{3!} \epsilon_{abc} E_i^a E_j^b E_k^c \epsilon^{ijk} \right|. \quad (116)$$

Therefore,

$$V_B = \int_B \sqrt{\left| \frac{1}{3!} \epsilon_{abc} E_i^a E_j^b E_k^c \epsilon^{ijk} \right|} d^3x \quad (117)$$

Following the similar regularization techniques as in the case of the area operator we write the previous integral as the limit of Riemann sums defined in terms of a decomposition of B in terms of three-cells (think of a cubic lattice for concreteness), then we quantize the regularized version^{21,22,49,50} using the fundamental flux operators of the previous section associated to infinitesimal two cells. Explicitly,

$$\widehat{V}_B = \lim_{N \rightarrow \infty} \widehat{V}_B^N \quad (118)$$

where

$$\widehat{V}_B^N = \sum_I^N \sqrt{\left| \frac{1}{3!} \epsilon_{abc} \widehat{E}_i(S_I^a) \widehat{E}_j(S_I^b) \widehat{E}_k(S_I^c) \epsilon^{ijk} \right|} \quad (119)$$

The way the surfaces S_I^a are chosen for a given three-cell I is illustrated on the

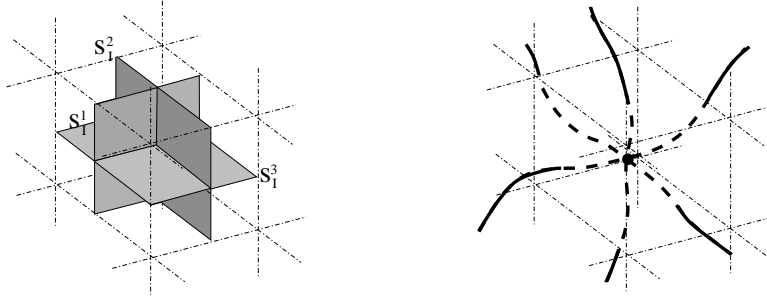


Fig. 8. Regularization of the volume operator. The infinitesimal two cells on which $E_i(S_I^a)$ are defined. The cellular decomposition is shrunk in the limiting process so that spin network nodes are contained in only one three cell.

left of Figure 8. The limit $N \rightarrow \infty$ is taken by keeping spin network nodes inside a single three-cell.

Let us finish this section by mentioning some general properties of the volume operator.

- (i) There are at least two consistent quantization of the volume operator. One is sensitive to the differential structure at the node⁵⁰ while the other is fully combinatorial²². The quantization of the scalar constraint first introduced by Thiemann⁵¹ uses the version of volume operators that is sensitive to the differential structure at the node.
- (ii) Three valent nodes are annihilated by the volume operator. This is a simple consequence of the Gauss constraint. The identity (107) implies that for a three valent node one can write one of the flux operators in (119) as a linear combination of the other two. The ϵ_{abc} in (119) makes the action of the operator equal to zero.
- (iii) The action of the volume operator vanishes on nodes whose edges lie on a plane, i.e., planar nodes.
- (iv) The spectrum of the volume is discrete. The eigenvalue problems is however more involved and an explicit closed formula is known only in special cases^{49,52,53,54}. Recently new manipulations have lead to simplifications in the spectral analysis of the volume operator⁵⁵.

4.2.4. *Geometric interpretation of spin network states*

The properties of the area and volume operator provide a very simple geometrical interpretation of spin network states. Edges of spin networks carry quanta of area while volume quantum numbers can be used to label nodes (plus some additional label to resolve degeneracy when needed). This interpretation is fully background independent and its combinatorial character implies that the geometric information stored in a quantum state of geometry is intrinsically diffeomorphism invariant. We can visualize a spin network as a polymer like excitation of space geometry consisting of volume excitations around spin network nodes connected by spin network links representing area excitations of the (dual) surfaces separating nodes. The picture is in complete agreement with the general direction in which a background independent formulation would be set up. The embedding or the coordinate system that we choose to draw the spin network graph does not carry any physical information. All the information about the degrees of freedom of geometry (hence the gravitational field) is contained in the combinatorial aspects of the graph (what is connected to what) and in the discrete quantum numbers labelling area quanta (spin labels of edges) and volume quanta (linear combinations of intertwiners at nodes). The fact that spin networks up to their embedding are physically relevant in LQG will become more clear in the next section, when we solve the diffeomorphism constraint.

4.2.5. *Continuum geometry*

After the discovery that geometric operators have discrete spectra an obvious question is whether the discreteness is compatible with the smooth geometry picture of the classical theory. One can in fact check that the spectrum crowds very rapidly

when one gets to larger geometries as the spacing between eigenvalues decreases exponentially for large eigenvalues.

4.3. Solutions of the diffeomorphism constraint: \mathcal{H}_{kin}^p and abstract spin networks

In Section 4.1.5 we solved the three quantum Einstein's equations defining the Hilbert space of $SU(2)$ gauge invariant states given by $\mathcal{H}_{kin}^g \subset \mathcal{H}_{kin}$. Now we turn our attention to the next three quantum constraint equations, the vector constraint (24). The technique that we apply to find the Hilbert space of diffeomorphism invariant states \mathcal{H}_{kin}^p is analogous to the one used to obtain \mathcal{H}_{kin}^g . However, because the orbits of the diffeomorphisms are not compact, diffeomorphism invariant states are not contained in the original \mathcal{H}_{kin} . In relation to \mathcal{H}_{kin} , they have to be regarded as distributional states⁴⁰.

A simple example will allow us to introduce the basic ideas. Consider an ordinary particle quantum system defined on a cylinder. Assume the kinematical Hilbert space is given by square integrable functions $\psi(\theta, z)$ or $\mathcal{H}_{kin} = \mathcal{L}^2(S^1 \times \mathbb{R})$. In addition assume one has to solve the following constraint equations:

$$\hat{p}_\theta \psi = 0 \quad \text{and} \quad \hat{p}_z \psi = 0. \quad (120)$$

The first constraint generates rotations around the z -axis, i.e. it has compact orbits, and, in this sense, is the analog of the Gauss constraint in LQG. The solutions of the first constraint are wave functions invariant under rotations around z , those that do not depend on θ . Therefore they are contained in the original Hilbert space \mathcal{H}_{kin} because, as the orbits of p_θ are compact, square integrable functions $\psi(z)$ (i.e., independent of θ) exist. The second constraint has non compact orbits and in this sense is the analog of the diffeomorphism constraint in LQG. The solution of the second constraint are functions that do not depend on z . They cannot be contained in \mathcal{H}_{kin} , as functions $\psi(\theta)$ cannot be in $\mathcal{H}_{kin} = \mathcal{L}^2(S^1 \times \mathbb{R})$. However given a suitable dense subset of $\Phi \subset \mathcal{H}_{kin}$ of test functions, e.g. functions of compact support, then any solution of the latter constraint can be given a meaning as a distribution. For instance a solution of both constraints does not depend neither on θ nor on z . Its wave function corresponds to a constant $(\psi_0|\theta, z \rangle = c$. The solution $(\psi_0|$ is clearly not in \mathcal{H}_{kin} . We use a rounded brackets in the notation to recall this fact. $(\psi_0|$ is in Φ^* , the topological dual of Φ , i.e., the set of linear functionals from Φ to \mathbb{C} . Its action on any arbitrary function of compact support $|\alpha \rangle \in \Phi \subset \mathcal{H}_{kin}$ is given by

$$(\psi_0|\alpha \rangle = \int dz d\theta c \alpha(\theta, z), \quad (121)$$

which is well defined. The action of $(\psi_0|$ extracts the gauge invariant information from the non gauge invariant state $|\alpha \rangle$. As vector spaces we have the relation $\Phi \subset \mathcal{H}_{kin} \subset \Phi^*$, usually called the Gelfand triple. In the case of LQG diffeomorphism invariant states are in the dual of the cylindrical functions Cyl. The Gelfand triple of

interest is $\text{Cyl} \subset \mathcal{H}_{kin} \subset \text{Cyl}^*$. Diffeomorphism invariant states have a well defined meaning as linear forms in Cyl^* .

Let us now apply the same idea to define diffeomorphism invariant states. Diffeomorphism transformations are easily represented in \mathcal{H}_{kin} . We denote $\mathcal{U}_{\mathcal{D}}[\phi]$ the operator representing the action of a diffeomorphism $\phi \in \text{Diff}(\Sigma)$. Its action can be defined directly on the dense subset of cylindrical functions $\text{Cyl} \subset \mathcal{H}_{kin}$. Given $\psi_{\gamma,f} \in \text{Cyl}$ as in (73) we have

$$\mathcal{U}_{\mathcal{D}}[\phi]\psi_{\gamma,f}[A] = \psi_{\phi^{-1}\gamma,f}[A], \quad (122)$$

which naturally follows from (71). Diffeomorphisms act on elements of Cyl (such as spin networks) by simply modifying the underlying graph in the obvious manner. Notice that $\mathcal{U}_{\mathcal{D}}[\phi]$ is unitary according to the definition (82).

Notice that because (122) is not weakly continuous there is no well defined notion of self-adjoint generator of infinitesimal diffeomorphisms in \mathcal{H}_{kin} ⁿ. In other words the unitary operator that implements a diffeomorphism transformation is well defined but there is no corresponding self adjoint operator whose exponentiation leads to $\mathcal{U}_{\mathcal{D}}[\phi]$. Therefore, the diffeomorphism constraint cannot be quantized in the Ashtekar-Lewandowski representation. This is not really a problem as $\mathcal{U}_{\mathcal{D}}[\phi]$ is all we need in the quantum theory to look for diffeomorphism invariant states. In LQG one replaces the second set of formal equations in (63) by the well defined equivalent requirement

$$\mathcal{U}_{\mathcal{D}}[\phi]\psi = \psi \quad (123)$$

for (distributional states) $\psi \in \text{Cyl}^*$.

We are now ready to explicitly write the solutions, namely

$$([\psi_{\gamma,f}]) = \sum_{\phi \in \text{Diff}(\Sigma)} \langle \psi_{\gamma,f} | \mathcal{U}_{\mathcal{D}}[\phi] = \sum_{\phi \in \text{Diff}(\Sigma)} \langle \psi_{\phi\gamma,f} |, \quad (124)$$

were the sum is over all diffeomorphisms. The brackets in $([\psi_{\gamma,f}])$ denote that the distributional state depends only on the equivalence class $[\psi_{\gamma,f}]$ under diffeomorphisms. Clearly we have $([\psi_{\gamma,f}] | \mathcal{U}_{\mathcal{D}}[\alpha] = ([\psi_{\gamma,f}] |$ for any $\alpha \in \text{Diff}(\Sigma)$. Now we need to check that $([\psi_{\gamma,f}]) \in \text{Cyl}^*$ so that the huge sum in (124) gives a finite result when applied to an element $|\psi_{\gamma',g}\rangle \in \text{Cyl}$, i.e., it is a well defined linear form. That this is the case follows from (82) (see the remark in Footnote n) as in

$$([\psi_{\gamma,f}] | \psi_{\gamma',g}\rangle \quad (125)$$

only a finite number of terms from (124) contribute. In fact for spin networks with no discrete symmetries there is only one non trivial contribution.

The action of $([\psi_{\gamma,f}] |$ is diffeomorphism invariant, namely

$$([\psi_{\gamma,f}] | \psi_{\gamma',g}\rangle = ([\psi_{\gamma,f}] | \mathcal{U}_{\mathcal{D}}[\phi] \psi_{\gamma',g}\rangle \quad (126)$$

ⁿ This is because for any $\phi \in \text{Diff}(\Sigma)$ the state $\mathcal{U}_{\mathcal{D}}[\phi]\psi$ is orthogonal to ψ for a generic $\psi \in \mathcal{H}_{kin}$.

The inner product \langle , \rangle_{diff} needed to promote the set of diffeomorphism invariant states to the Hilbert space \mathcal{H}_{kin}^D is defined as

$$\langle [\psi_{\gamma,f}] | [\psi_{\gamma',g}] \rangle_{diff} = ([\psi_{\gamma,f}] | \psi_{\gamma',g} \rangle \quad (127)$$

Due to (126) the previous equation is well defined among diffeomorphism equivalence classes of states under the action of diffeomorphisms and hence this is denoted by the brackets on both sides.

4.3.1. The quantization of the scalar constraint

In this section we sketch the regularization and definition of the last constraint to be quantized and solved: the scalar constraint. The smeared version of the classical scalar constraint is

$$\begin{aligned} S(N) &= \int_{\Sigma} dx^3 N \frac{E_i^a E_j^b}{\sqrt{\det(E)}} \left(\epsilon^{ij}{}^k F_{ab}^k - 2(1 + \gamma^2) K_{[a}^i K_{b]}^j \right) = \\ &= S^E(N) - 2(1 + \gamma^2) T(N), \end{aligned} \quad (128)$$

where $K_a^i = \gamma^{-1}(A_a^i - \Gamma_a^i)$, and in the second line we have introduced a convenient separation of the constraint into what is called the Euclidean contribution $S^E(N)$ given by

$$S^E(N) = \int_{\Sigma} dx^3 N \frac{E_i^a E_j^b}{\sqrt{\det(E)}} \epsilon^{ij}{}^k F_{ab}^k, \quad (129)$$

and the extra piece

$$T(N) = \int_{\Sigma} dx^3 N \frac{E_i^a E_j^b}{\sqrt{\det(E)}} K_{[a}^i K_{b]}^j. \quad (130)$$

The terms in the constraint look in fact very complicated. On the one hand they are highly non linear which anticipates difficulties in the quantization related to regularization issues and potential UV divergences, factor ordering ambiguities, etc. For instance the factor $1/\det(E)$ looks quite complicated at first sight as does the spin connection Γ_a^i in the expression of K_a^i (recall its definition (19) and (20) in terms of the basic triad variables).

The crucial simplification of the apparently intractable problem came from the ideas of Thiemann⁵⁶. He observed that if one introduces the phase space functional

$$\bar{K} := \int_{\Sigma} K_a^i E_i^a, \quad (131)$$

then the following series of identities hold:

$$K_a^i = \gamma^{-1}(A_a^i - \Gamma_a^i) = \frac{1}{\kappa\gamma} \{A_a^i, \bar{K}\}, \quad (132)$$

$$\bar{K} = \frac{1}{\gamma^{3/2}} \{S^E(1), V\}, \quad (133)$$

where $V = \int \sqrt{\det(E)}$ is the volume of Σ , and finally

$$\frac{E_i^b E_j^c}{\sqrt{\det(E)}} \epsilon^{ijk} \epsilon_{abc} = \frac{4}{\kappa \gamma} \{A_a^k, V\}. \quad (134)$$

With all this we can write the terms in the scalar constraint by means of Poisson brackets among quantities that are simple enough to consider their quantization. The Euclidean constraint can be written as

$$S^E(N) = \int_{\Sigma} dx^3 N \epsilon^{abc} \delta_{ij} F_{ab}^i \{A_c^j, V\}, \quad (135)$$

while the term $T(N)$ becomes

$$T(N) = \int_{\Sigma} dx^3 \frac{N}{\kappa^2 \gamma^3} \epsilon^{abc} \epsilon_{ijk} \{A_a^i, \{S^E(1), V\}\} \{A_b^j, \{S^E(1), V\}\} \{A_c^k, V\}. \quad (136)$$

The new form suggests that we can quantize the constraint by promoting the argument of the Poisson brackets to operators and the Poisson brackets them self to commutators in the standard way. One needs the volume operator V , whose quantization was already discussed, and the quantization of the connection and curvature. We present here the basic idea behind the quantization of these. For a precise treatment the reader is encouraged to read Thiemann's original work^{51,57} and book²⁰. Given an infinitesimal loop α_{ab} on the ab -plane with coordinate area ϵ^2 , the curvature tensor can be regularized observing that

$$h_{\alpha_{ab}}[A] - h_{\alpha_{ab}}^{-1}[A] = \epsilon^2 F_{ab}^i \tau_i + \mathcal{O}(\epsilon^4). \quad (137)$$

Similarly the Poisson bracket $\{A_a^i, V\}$ is regularized as

$$h_{e_a}^{-1}[A] \{h_{e_a}[A], V\} = \epsilon \{A_a^i, V\} + \mathcal{O}(\epsilon^2), \quad (138)$$

where e_a is a path along the a -coordinate of coordinate length ϵ . With this we can write

$$\begin{aligned} S^E(N) &= \\ &= \lim_{\epsilon \rightarrow 0} \sum_I N_I \epsilon^3 \epsilon^{abc} \text{Tr} [F_{ab}(A) \{A_c, V\}] = \\ &= \lim_{\epsilon \rightarrow 0} \sum_I N_I \epsilon^{abc} \text{Tr} \left[(h_{\alpha_{ab}^I}[A] - h_{\alpha_{ab}^I}^{-1}[A]) h_{e_c^I}^{-1}[A] \{h_{e_c^I}[A], V\} \right], \end{aligned} \quad (139)$$

where in the first equality we have replaced the integral (135) by a Riemann sum over cells of coordinate volume ϵ^3 and in the second line we have written it in terms of holonomies. Notice that the explicit dependence on the cell size ϵ has disappeared in the last line. The cells are labelled with the index I in analogy to

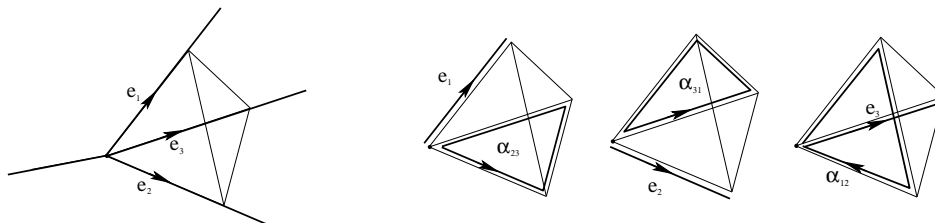


Fig. 9. A tetrahedral infinitesimal cell adapted to a four valent spin network node. On the right: the three non trivial contributions of the cell to the regularized expression (140).

the regularization used for the area and volume in previous sections. The loop α_{ab}^I is an infinitesimal closed loop of coordinate area ϵ^2 in the ab -plane associated to the I -th cell, while the edge e_a^I is the corresponding edge of coordinate length ϵ dual to the ab -plane (see Figure 9 for a cartoon of the regularization first introduced by Thiemann). The idea now is to promote this regulated expression to an operator by quantizing the ingredients of the formula: notice that we already know how to do that as the expression involves the holonomies and the volume, both well defined operators in \mathcal{H}_{kin} . The quantum constraint can formally be written as

$$\widehat{S}^E(N) = \lim_{\epsilon \rightarrow 0} \sum_I N_I \epsilon^{abc} \text{Tr} \left[\widehat{h}_{\alpha_{ab}^I} [A] - \widehat{h}_{\alpha_{ab}^I}^{-1} [A] \widehat{h}_{e_c^I}^{-1} [A] \left\{ \widehat{h}_{e_c^I} [A], \widehat{V} \right\} \right]. \quad (140)$$

Now in order to have a rigorous definition of $\widehat{S}^E(N)$ one needs to show that the previous limit exists in the appropriate Hilbert space.

It is useful to describe some of the qualitative features of the argument of the limit in (140) which we refer to as the *regulated quantum scalar constraint* and we denote $\widehat{S}_\epsilon(N)$. It is easy to see that the regulated quantum scalar constraint acts only on spin network nodes, namely

$$\widehat{S}_\epsilon(N) \psi_{\gamma,f} = \sum_{n \in \gamma} N_n \widehat{S}_\epsilon^n \psi_{\gamma,f}, \quad (141)$$

where \widehat{S}_ϵ^n acts only on the node $n \subset \gamma$ and N_n is the value of the lapse $N(x)$ at the node. This is a simple consequence of the very same property of the volume operator (118). Due to the action of the infinitesimal loop operators representing the regularized curvature, the scalar constraint modifies spin networks by creating new links around nodes whose amplitudes depend on the details of the action of the volume operator, the local spin labels and other local features at the nodes. If we concentrate on the Euclidean constraint, for simplicity, its action on four valent nodes can be written as

$$\widehat{S}_\epsilon^n = \sum_{op} S_{jklm,opq} + \sum_{op} S_{jlmk,opq} + \sum_{op} S_{jmk l,opq} \quad (142)$$

where $q = 1/2$ in the case of (140) (we will see in the sequel that q can be any arbitrary spin: this is one of the quantization ambiguities in the theory), and $S_{jklm,opq}$ are coefficients that are independent of ϵ and can be computed explicitly from (140).

Now we analyze the removal of the regulator. Since the only dependence of ϵ is in the position of the extra link in the resulting spin network states, the limit $\epsilon \rightarrow 0$ can be defined in the Hilbert space of diffeomorphism invariant states \mathcal{H}_{kin}^p . The key property is that in the diffeomorphism invariant context the position of the new link in (142) is irrelevant. Therefore, given a diffeomorphism invariant state $([\phi] \in \mathcal{H}_{kin}^p \in \text{Cyl}^*$, as defined as in (124), the quantity $(\phi|\widehat{S}_\epsilon(N)|\psi \rangle$ is well defined and independent of ϵ . In other words the limit

$$(\phi|\widehat{S}(N)|\psi \rangle = \lim_{\epsilon \rightarrow 0} (\phi|\widehat{S}_\epsilon(N)|\psi \rangle \quad (143)$$

exists trivially for any given $\psi \in \mathcal{H}_{kin}$. A careful analysis on how this limit is defined can be found in²⁰.

An important property of the definition of the quantum scalar constraint is that the new edges added (or annihilated) are of a very special character. For instance, not only do the new nodes in (142) carry zero volume but also they are invisible to the action of the quantum scalar constraint. The reason for that is that the new three-valent nodes are planar and therefore the action of the commutator of the holonomy with the volume operator in (140) vanishes identically (recall the properties of the volume operator at the end of Section 4.2). For this reason it is useful to refer to these edges as *exceptional edges*.

This property of Thiemann's constraint is indeed very important for the consistency of the quantization. The non trivial consistency condition on the quantization of the scalar constraint corresponds to the quantum version of (41). The correct commutator algebra is satisfied in the sense that for diffeomorphism invariant states $(\phi \in \mathcal{H}_{kin}^p$ (defined as in (124))

$$(\phi|[\widehat{S}(N), \widehat{S}(M)]\psi \rangle = 0, \quad (144)$$

for any $\psi \in \mathcal{H}_{kin}^q$. The l.h.s. vanishes due to the special property of exceptional edges as can be checked by direct calculation. Notice that r.h.s. of (41) is expected to annihilate elements of \mathcal{H}_{kin}^p —at least for the appropriate factor ordering— so

that the previous equation is in agreement with (41) and the quantization is said to be *anomaly-free*⁵⁸. All known^{59,60,61} consistent quantizations satisfy the Abelian property (144) in $\mathcal{H}_{kin}^{\mathcal{P}}$. We will come back to this issue in the next Subsection.

Notice that (134) is the co-triad e_a^i according to (20); this opens the way for the quantization of the metric $q_{ab} = e_a^i e_b^j \delta_{ij}$ that is necessary for the inclusion of matter. A well defined quantization of the scalar constraint including Yang-Mills fields, scalar fields and fermions has been put forward by Thiemann⁶².

4.3.2. Solutions of the scalar constraint, physical observables, difficulties

Here we briefly explore some of the generic consequences of the theory constructed so far. The successful definition of the quantum scalar constraint operator including the cases with realistic matter couplings is a remarkable achievement of loop quantum gravity. There are however some issues that we would like to emphasize here.

There is a large degree of ambiguity on the definition of the quantum scalar constraint. The nature of solutions or the dynamics seems to depend critically on these ambiguities. For instance it is possible to arrive at a completely consistent quantization by essentially replacing the holonomies in (140)—defined in the fundamental representation of $SU(2)$ —by the corresponding quantities evaluated on an arbitrary representation⁵⁹. In the applications of the theory to simple systems such as in loop quantum cosmology this is known to have an important physical effect⁶³. Ambiguities are also present in the way in which the paths defining the holonomies that regularize the connection A_a^i and the curvature $F_{ab}^i(A)$ in (140) are chosen. See for instance Section C in¹⁹ for an alternative to Thiemann’s prescription and a discussion of the degree of ambiguity involved. There are factor ordering ambiguities as well, which is evident from (140). Therefore instead of a single theory we have infinitely many theories which are mathematically consistent. A yet unresolved issue is whether any of these theories is rich enough to reproduce general relativity in the classical continuum limit.

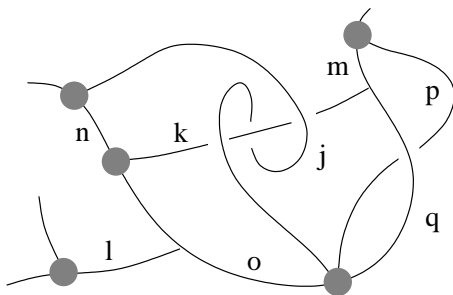


Fig. 10. Solutions of the scalar constraint as dressed (diff-invariant) spin networks.

All the known consistent quantizations of the scalar constraint satisfy property (144). Quantizations that satisfy this property seem to share a property that is often

the scalar constraint satisfying (144)^P. However, at present it is not very clear what the physical interpretation of these observables would be.

On the other hand, the ultra-local character of the solutions has raised some concerns of whether the quantum theories of the Thiemann type can reproduce the physics of gravity in the classical limit⁶⁴. In order to illustrate this point let us go back to the classical theory. Since at the classical level we can invert the transformations that lead to new variables in Section 2 let us work with the constraints in ADM variables. We simplify the discussion by considering the York-Lichnerowicz conformal decompositions for initial data. For the discussion here we can specialize to the time-symmetric case $K_{ab} = 0$ and we take the ansatz $q_{ab} = \psi^4 q_{ab}^0$ for some given q_{ab}^0 defined up to a conformal factor. The vector constraint is in this case identically zero and the only non trivial constraint is the scalar constraint that can be shown to be

$$\Delta^0 \psi - \frac{1}{8} R(q^0) = 0, \quad (146)$$

where Δ^0 is the covariant Laplacian defined with respect to q_{ab}^0 . The point is that the previous equation is manifestly *elliptic* which is a general property of the scalar constraint written in this form⁶⁷. This means that if we give the value of ψ on a sphere, the scalar constraint (146) will determine the value of ψ inside. The scalar constraint in general relativity imposes a condition among unphysical degrees of freedom (represented by ψ in this case) that “propagate” along the initial value surface⁹. The only point in writing the initial value problem in this way is to emphasize this property of the scalar constraint.

Coming back to the quantum theory one can construct semi-classical states by taking linear combinations of kinematical spin-network states in order to approximate some classical geometry q_{ab} ^{68,69}. In that context one could define a sphere with some given semi-classical area. Now because the scalar constraint acts only in the immediate vicinity of nodes and does not change the value of the spins of the edges that connect different *dressed* nodes it is not clear how the elliptic character of the classical scalar constraint would be recovered in this semi-classical context. In other words, how is it that quantizations of the scalar constraint that are ultra-local in the sense above can impose conditions restricting unphysical degrees of freedom in the interior of the sphere once boundary conditions defining the geometry of the

^PThe intuitive idea here is presented in terms of a given solution of the constraint based on a given family of graphs: the dressed spin-networks. One should keep in mind that defining a quantum operators representing a Dirac observable implies defining its action on the whole \mathcal{H}_{phys} . A simple example of Dirac operator whose action is defined for all solutions is for instance $O_D = \sum_e j_e$, i.e., the sum of the spins connecting dressed nodes⁶⁶.

⁹With the time-symmetric ansatz $K_{ab} = 0$ for the five degrees of freedom in the conformal metric q_{ab}^0 only one local physical degree of freedom remains after factoring out the action of the constraints as generating functions of gauge transformations. In this context our truncation leads to only one free data per space-point. Despite these restrictions there are many radiating spacetimes and other interesting solutions in this sector.

sphere are given? Because we still know very little about the semi-classical limit these concerns should be taken as open issues that deserve a more precise analysis.

Motivated by these concerns different avenues of research have been explored with the hope of finding alternatives and some guiding principles that would lead to a clearer understanding of the physics behind the quantum scalar constraint. For instance, the previous concerns have led to the exploration of the formalism of consistent discretizations presented in these lectures by Rodolfo Gambini^{70,71,72}. The spin foam approach has been motivated to a large extent by the hope of solving the issue of ambiguities and ultra-locality from a covariant perspective as well as by the search of a systematic definition of the physical scalar product. An alternative strategy to the quantization of the scalar constraint and the construction of the physical inner product that in essence circumvents the anomaly freeness condition (144) has been recently proposed by Thiemann⁷³.

4.4. An important application: computation of black hole entropy

In the actual talks we described the main ideas behind the computation of black hole entropy in LQG. The philosophy was to quantize a sector of the theory containing an isolated horizon⁷⁴ and then to count the number of physical states \mathcal{N} compatible with a given macroscopic area a_0 of the horizon. The entropy S of the black hole is defined by $S = \ln(\mathcal{N})$. The counting can be made exactly when $a_0 \gg \ell_p^2$. The result is

$$S = \frac{\gamma_0}{\gamma} \frac{a_0}{4\ell_p^2} + \mathcal{O}(\ln(\frac{a_0}{\ell_p^2})), \quad (147)$$

where the real number $\gamma_0 = 0.2375\dots$ follows from the counting^{75,76}. The value of the Immirzi parameter comes from the fact that γ appears as a pre-factor in the spectrum of the area operator. It is important to emphasize that the computation of S is independent of the details of the quantization of the scalar constraint. Semi-classical considerations lead to $S = a_0/(4\ell_p^2)$, the computation above can be used to fix the value of the Immirzi parameter, namely

$$\gamma = \gamma_0. \quad (148)$$

The above computation can be performed for any black hole of the Kerr-Newman family and the result is consistent with the chosen value of γ .

The reader interested in the details of this calculation is encouraged to study the original papers^{77,78,79,80,81}, the recent review by Ashtekar and Lewandowski, or the book of Rovelli⁴. Attention is drawn to the recent results of Domagala and Lewandowski⁷⁵ and Meissner⁷⁶.

5. Spin Foams: the path integral representation of the dynamics in loop quantum gravity

The spin foam approach was motivated by the need to shed new light on the issue of the dynamics of loop quantum gravity by attempting the construction of the path

integral representation of the theory. In this section we will introduce the main ideas behind the approach by considering simpler systems in some detail. For a broader overview see the review articles^{82,83} and references therein.

The solutions of the scalar constraint can be characterized by the definition of the generalized projection operator P from the kinematical Hilbert space \mathcal{H}_{kin} into the kernel of the scalar constraint \mathcal{H}_{phys} . Formally one can write P as

$$P = \left\langle \prod_{x \subset \Sigma} \delta(\widehat{S}(x)) \right\rangle = \int D[N] \exp \left[i \int_{\Sigma} N(x) \widehat{S}(x) \right]. \quad (149)$$

A formal argument^{84,85} shows that P can also be defined in a manifestly covariant manner as a regularization of the formal path integral of general relativity (in 2d gravity this is shown in⁸⁶, here we will show that this is the case in 3d gravity). In first order variables it becomes

$$P = \int D[e] D[A] \mu[A, e] \exp [i S_{GR}(e, A)] \quad (150)$$

where e is the tetrad field, A is the spacetime connection, and $\mu[A, e]$ denotes the appropriate measure.

In both cases, P characterizes the space of solutions of *quantum Einstein equations* as for any arbitrary state $|\phi\rangle \in \mathcal{H}_{kin}$ then $P|\phi\rangle$ is a (formal) solution of (63). Moreover, the matrix elements of P define the physical inner product $\langle \cdot, \cdot \rangle_p$ providing the vector space of solutions of (63) with the Hilbert space structure that defines \mathcal{H}_{phys} . Explicitly

$$\langle s, s' \rangle_p := \langle P s, s' \rangle,$$

for $s, s' \in \mathcal{H}_{kin}$.

When these matrix element are computed in the *spin network* basis (see Section 4.1.5), they can be expressed as a sum over amplitudes of '*spin network histories*': *spin foams* (Figure 11). The latter are naturally given by foam-like combinatorial structures whose basic elements carry quantum numbers of geometry (see Section 4.2). A *spin foam* history⁸⁷, from the state $|s\rangle$ to the state $|s'\rangle$, is denoted by a pair $(F_{s \rightarrow s'}, \{j\})$ where $F_{s \rightarrow s'}$ is the 2-complex with boundary given by the graphs of the *spin network* states $|s'\rangle$ and $|s\rangle$ respectively, and $\{j\}$ is the set of spin quantum numbers labelling its edges (denoted $e \subset F_{s \rightarrow s'}$) and faces (denoted $f \subset F_{s \rightarrow s'}$). Vertices are denoted $v \subset F_{s \rightarrow s'}$. The physical inner product can be expressed as a sum over *spin foam* amplitudes

$$\begin{aligned} \langle s', s \rangle_p &= \langle P s', s \rangle = \\ & \sum_{F_{s \rightarrow s'}} N(F_{s \rightarrow s'}) \sum_{\{j\}} \prod_{f \subset F_{s \rightarrow s'}} A_f(j_f) \prod_{e \subset F_{s \rightarrow s'}} A_e(j_e) \prod_{v \subset F_{s \rightarrow s'}} A_v(j_v), \end{aligned} \quad (151)$$

where $N(F_{s \rightarrow s'})$ is a (possible) normalization factor, and $A_f(j_f)$, $A_e(j_e)$, and $A_v(j_v)$ are the 2-cell or face amplitude, the edge or 1-cell amplitude, and the 0-cell of

vertex amplitude respectively. These local amplitudes depend on the spin quantum numbers labelling neighboring cells in $F_{s \rightarrow s'}$ (e.g. the vertex amplitude of the vertex magnified in Figure 11 is $A_v(j, k, l, m, n, s)$).

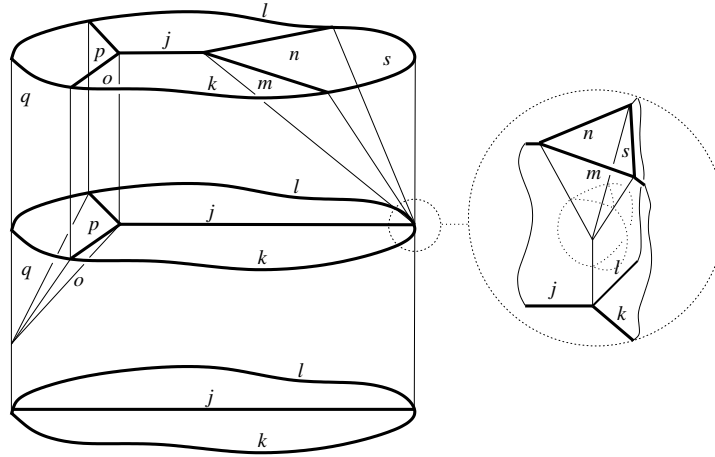


Fig. 11. A *spin foam* as the ‘colored’ 2-complex representing the transition between three different *spin network* states. A transition vertex is magnified on the right.

The underlying discreteness discovered in LQG is crucial: in the *spin foam* representation, the functional integral for gravity is replaced by a sum over amplitudes of combinatorial objects given by foam-like configurations (*spin foams*) as in (151). A *spin foam* represents a possible history of the gravitational field and can be interpreted as a set of transitions through different quantum states of space. Boundary data in the path integral are given by the polymer-like excitations (*spin network* states, Figure 5) representing 3-geometry states in LQG.

6. *Spin foams* in 3d quantum gravity

We introduce the concept of *spin foams* in a more explicit way in the context of the quantization of three dimensional Riemannian gravity. In Section 6.2 and 6.3 we will present the definition of P from the canonical and covariant view point formally stated in the introduction by Equations (149) and (150) respectively^r. For other approaches to 3d quantum gravity see the book of Carlip⁹¹.

^rIt is well known that the physical inner product for 3d Riemannian gravity can be defined using group averaging techniques⁸⁸, here we review this and use the approach to introduce the spin foam representation^{89,90}.

6.1. The classical theory

Riemannian gravity in 3 dimensions is a theory with no local degrees of freedom, i.e., a topological theory. Its action (in the first order formalism) is given by

$$S(e, \omega) = \int_M \text{Tr}(e \wedge F(\omega)), \quad (152)$$

where $M = \Sigma \times \mathbb{R}$ (for Σ an arbitrary Riemann surface), ω is an $SU(2)$ -connection and the triad e is an $su(2)$ -valued 1-form. The gauge symmetries of the action are the local $SU(2)$ gauge transformations

$$\delta e = [e, \alpha], \quad \delta \omega = d_\omega \alpha, \quad (153)$$

where α is a $su(2)$ -valued 0-form, and the ‘topological’ gauge transformation

$$\delta e = d_\omega \eta, \quad \delta \omega = 0, \quad (154)$$

where d_ω denotes the covariant exterior derivative and η is a $su(2)$ -valued 0-form. The first invariance is manifest from the form of the action, while the second is a consequence of the Bianchi identity, $d_\omega F(\omega) = 0$. The gauge symmetries are so large that all the solutions to the equations of motion are locally pure gauge. The theory has only global or topological degrees of freedom.

Upon the standard 2+1 decomposition, the phase space in these variables is parametrized by the pull back to Σ of ω and e . In local coordinates one can express them in terms of the 2-dimensional connection A_a^i and the triad field $E_j^b = \epsilon^{bc} e_c^k \eta_{jk}$ where $a = 1, 2$ are space coordinate indices and $i, j = 1, 2, 3$ are $su(2)$ indices. The symplectic structure is defined by

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta^{(2)}(x, y). \quad (155)$$

Local symmetries of the theory are generated by the first class constraints

$$D_b E_j^b = 0, \quad F_{ab}^i(A) = 0, \quad (156)$$

which are referred to as the Gauss law and the curvature constraint respectively—the quantization of these is the analog of (63) in 4d. This simple theory has been quantized in various ways in the literature⁹¹, here we will use it to introduce the *spin foam* quantization.

6.2. Spin foams from the Hamiltonian formulation

The physical Hilbert space, \mathcal{H}_{phys} , is defined by those ‘states’ in \mathcal{H}_{kin} that are annihilated by the constraints. As we discussed in Section 4.1.5, spin network states solve the Gauss constraint— $\widehat{D}_a E_i^a |s\rangle = 0$ —as they are manifestly $SU(2)$ gauge invariant. To complete the quantization one needs to characterize the space of solutions of the quantum curvature constraints (\widehat{F}_{ab}^i), and to provide it with the physical inner product. The existence of \mathcal{H}_{phys} is granted by the following

Theorem 1. There exists a normalized positive linear form P over Cyl , i.e.

$P(\psi^*\psi) \geq 0$ for $\psi \in \text{Cyl}$ and $P(1) = 1$, yielding (through the GNS construction⁴⁵) the physical Hilbert space \mathcal{H}_{phys} and the physical representation π_p of Cyl .

The state P contains a very large Gelfand ideal (set of zero norm states) $J := \{\alpha \in \text{Cyl} \text{ s.t. } P(\alpha^*\alpha) = 0\}$. In fact the physical Hilbert space $\mathcal{H}_{phys} := \overline{\text{Cyl}/J}$ corresponds to the quantization of finitely many degrees of freedom. This is expected in 3d gravity as the theory does not have local excitations (no ‘gravitons’). The representation π_p of Cyl solves the curvature constraint in the sense that for any functional $f_\gamma[A] \in \text{Cyl}$ defined on the sub-algebra of functionals defined on contractible graphs $\gamma \subset \Sigma$, one has that

$$\pi_p[f_\gamma]\Psi = f_\gamma[0]\Psi. \quad (157)$$

This equation expresses the fact that ‘ $\widehat{F} = 0$ ’ in \mathcal{H}_{phys} (for flat connections parallel transport is trivial around a contractible region). For $s, s' \in \mathcal{H}_{kin}$, the physical inner product is given by

$$\langle s, s' \rangle_p := P(s^*s), \quad (158)$$

where the $*$ -operation and the product are defined in Cyl .

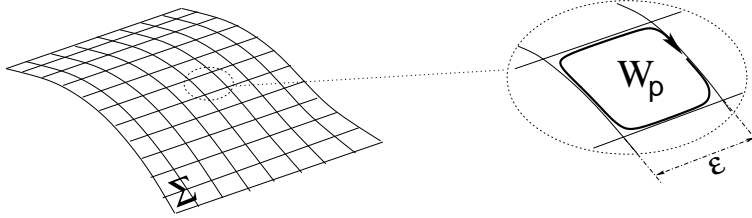


Fig. 12. Cellular decomposition of the space manifold Σ (a square lattice in this example), and the infinitesimal plaquette holonomy $W_p[A]$.

The previous equation admits a ‘sum over histories’ representation⁸⁹. We shall introduce the concept of the *spin foam* representation as an explicit construction of the positive linear form P which, as in (149), is formally given by

$$P = \int D[N] \exp(i \int_{\Sigma} \text{Tr}[N \widehat{F}(A)]) = \prod_{x \subset \Sigma} \delta[\widehat{F}(A)], \quad (159)$$

where $N(x) \in \mathfrak{su}(2)$. One can make the previous formal expression a rigorous definition if one introduces a regularization. Given a partition of Σ in terms of 2-dimensional plaquettes of coordinate area ϵ^2 one has that

$$\int_{\Sigma} \text{Tr}[NF(A)] = \lim_{\epsilon \rightarrow 0} \sum_{p^i} \epsilon^2 \text{Tr}[N_{p^i} F_{p^i}], \quad (160)$$

where N_{p^i} and F_{p^i} are values of N^i and $\epsilon^{ab} F_{ab}^i[A]$ at some interior point of the plaquette p^i and ϵ^{ab} is the Levi-Civita tensor. Similarly the holonomy $W_{p^i}[A]$ around

the boundary of the plaquette p^i (see Figure 12) is given by

$$W_{p^i}[A] = \mathbb{1} + \epsilon^2 F_{p^i}(A) + \mathcal{O}(\epsilon^2), \quad (161)$$

where $F_{p^i} = \tau_j \epsilon^{ab} F_{ab}^j(x_{p^i})$ (τ_j are the generators of $su(2)$ in the fundamental representation). The previous two equations lead to the following definition: given $s \in \text{Cyl}$ (think of *spin network* state based on a graph γ) the linear form $P(s)$ is defined as

$$P(s) := \lim_{\epsilon \rightarrow 0} \langle \Omega \prod_{p^i} \int dN_{p^i} \exp(i\text{Tr}[N_{p^i} W_{p^i}]), s \rangle. \quad (162)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the AL-representation and $|\Omega\rangle$ is the ‘vacuum’ ($1 \in \text{Cyl}$) in the AL-representation. The partition is chosen so that the links of the underlying graph γ border the plaquettes. One can easily perform the integration over the N_{p^i} using the identity (Peter-Weyl theorem)

$$\int dN \exp(i\text{Tr}[NW]) = \sum_j (2j+1) \text{Tr}[\overset{j}{\Pi}(W)]. \quad (163)$$

Using the previous equation

$$P(s) := \lim_{\epsilon \rightarrow 0} \prod_{p^i} \sum_{j(p^i)} (2j(p^i) + 1) \langle \Omega \text{Tr}[\overset{j(p^i)}{\Pi}(W_{p^i})], s \rangle, \quad (164)$$

where $j(p^i)$ is the spin labelling elements of the sum (163) associated to the i th plaquette. Since the $\text{Tr}[\overset{j}{\Pi}(W)]$ commute the ordering of plaquette-operators in the previous product does not matter. It can be shown that the limit $\epsilon \rightarrow 0$ exists and one can give a closed expression of $P(s)$.

Some remarks are in order:

Remark 1: The argument of the limit in (164) satisfies the following inequalities

$$\begin{aligned} & \left| \sum_{j(p^i)} (2j(p^i) + 1) \mu_{AL} \left(\prod_{p^i} \chi_{j(p^i)}(W_{p^i}[A]) \overline{s[A]} s'[A] \right) \right| \leq \\ & \leq C \left| \sum_{j_{p^i}} (2j(p^i) + 1) \mu_{AL} \left(\prod_{p^i} \chi_{j(p^i)}(W_{p^i}[A]) \right) \right| = \\ & = C \sum_j (2j+1)^{2-2g}, \end{aligned} \quad (165)$$

where we have used (82), C is a real positive constant, and the last equation follows immediately from the definition of the Ashtekar-Lewandowski measure μ_{AL} ⁸⁶. The convergence of the sum for genus $g \geq 2$ follows directly.

Remark 2: The case of the sphere $g = 0$ is easy to regularize. In this case (164) diverges simply because of a redundancy in the product of delta distributions in the notation of (20). This is a consequence of the discrete analog of the Bianchi identity.

$$\text{Tr}[\prod^k(W_p)] \triangleright \left[\begin{array}{c} \text{---} \\ | \quad j \quad | \\ \text{---} \end{array} \text{p} \right] = \left[\begin{array}{c} \text{---} \\ | \quad j \quad | \\ \text{---} \end{array} \text{k} \right] = \sum_m N_{j,m,k} \left[\begin{array}{c} \text{---} \\ | \quad j \quad | \\ \text{---} \\ \text{---} \\ | \quad m \quad | \\ \text{---} \end{array} \text{k} \right]$$

Fig. 13. Graphical notation representing the action of one plaquette holonomy on a *spin network* state. On the right is the result written in terms of the *spin network* basis. The amplitude $N_{j,m,k}$ can be expressed in terms of Clebsch-Gordan coefficients.

It is easy to check that eliminating a single arbitrary plaquette holonomy from the product in (164) makes P well defined and produces the correct (one dimensional) \mathcal{H}_{phys} .

The case of the torus $g = 1$ is more subtle; in fact our prescription must be modified in that case⁹¹.

Remark 3: It is immediate to see that (164) satisfies hermitian condition

$$\langle Ps, s' \rangle = \overline{\langle Ps', s \rangle}. \quad (166)$$

Remark 4: The positivity condition also follows from the definition $\langle Ps, s \rangle \geq 0$.

Now in the AL-representation, each $\text{Tr}[\prod^{j(p^i)}(W_{p^i})]$ acts by creating a closed loop in the j_{p^i} representation at the boundary of the corresponding plaquette (Figures 13 and 15). One can introduce a (non-physical) time parameter that works simply as a coordinate providing the means of organizing the series of actions of plaquette loop-operators in (164); i.e., one assumes that each of the loop actions occur at different ‘times’. We have introduced an auxiliary time slicing (arbitrary parametrization). If one inserts the AL partition of unity

$$\mathbb{1} = \sum_{\gamma \subset \Sigma} \sum_{\{j\}_\gamma} |\gamma, \{j\} \rangle \langle \gamma, \{j\}|, \quad (167)$$

where the sum is over the complete basis of *spin network* states $\{|\gamma, \{j\} \rangle\}$ —based on all graphs $\gamma \subset \Sigma$ and with all possible spin labelling—between each time slice, one arrives at a sum over spin-network histories representation of $P(s)$. More precisely, $P(s)$ can be expressed as a sum over amplitudes corresponding to a series of transitions that can be viewed as the ‘time evolution’ between the ‘initial’ *spin network* s and the ‘final’ ‘vacuum state’ Ω . The physical inner product between *spin networks* s , and s' is defined as

$$\langle s, s' \rangle_p := P(s^* s'),$$

and can be expressed as a sum over amplitudes corresponding to transitions interpolating between the ‘initial’ *spin network* s' and the ‘final’ *spin network* s (e.g. Figures 14 and 16).

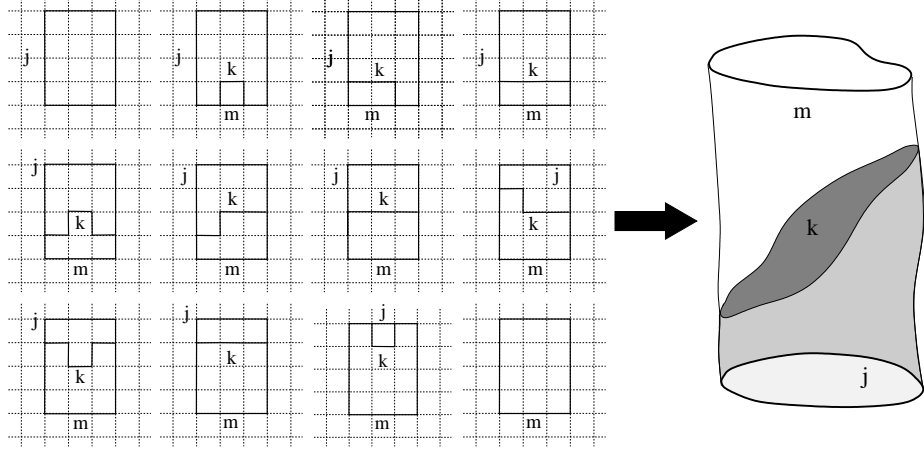
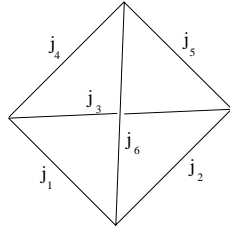


Fig. 14. A set of discrete transitions in the loop-to-loop physical inner product obtained by a series of transitions as in Figure 13. On the right, the continuous *spin foam* representation in the limit $\epsilon \rightarrow 0$.

$$\text{Tr}[\Pi^n(W_p)] \triangleright \begin{array}{c} j \quad k \\ \diagdown \quad \diagup \\ \text{---} P \text{---} \\ \diagup \quad \diagdown \\ m \end{array} = \begin{array}{c} j \quad k \\ \diagdown \quad \diagup \\ \text{---} n \text{---} \\ \diagup \quad \diagdown \\ m \end{array} = \sum_{o,p} \frac{1}{\Delta_n \Delta_j \Delta_k \Delta_m} \left\{ \begin{array}{ccc} j & k & m \\ n & o & p \end{array} \right\} \begin{array}{c} j \quad k \\ \diagdown \quad \diagup \\ \text{---} n \text{---} \\ \diagup \quad \diagdown \\ p \quad o \\ m \end{array}$$

Fig. 15. Graphical notation representing the action of one plaquette holonomy on a *spin network* vertex. The object in brackets ($\{\}$) is a $6j$ -symbol and $\Delta_j := 2j + 1$.

Spin network nodes evolve into edges while *spin network* links evolve into 2-dimensional faces. Edges inherit the intertwiners associated to the nodes and faces inherit the spins associated to links. Therefore, the series of transitions can be represented by a 2-complex whose 1-cells are labelled by intertwiners and whose 2-cells are labelled by spins. The places where the action of the plaquette loop operators create new links (Figures 15 and 16) define 0-cells or vertices. These foam-like structures are the so-called *spin foams*. The *spin foam* amplitudes are purely combinatorial and can be explicitly computed from the simple action of the loop operator in the AL-representation (Section 4.1.3). A particularly simple case arises when the *spin network* states s and s' have only 3-valent nodes. Explicitly

$$\langle s, s' \rangle_p := P(s^* s') = \sum_{\{j\}} \prod_{f \in F_{s \rightarrow s'}} (2j_f + 1)^{\frac{\nu_f}{2}} \prod_{v \in F_{s \rightarrow s'}} \text{tetrahedron}(j_1, j_2, j_3, j_4, j_5, j_6), \quad (168)$$


where the notation is that of (151), and $\nu_f = 0$ if $f \cap s \neq 0 \wedge f \cap s' \neq 0$, $\nu_f = 1$ if $f \cap s \neq 0 \vee f \cap s' \neq 0$, and $\nu_f = 2$ if $f \cap s = 0 \wedge f \cap s' = 0$. The tetrahedral diagram denotes a $6j$ -symbol: the amplitude obtained by means of the natural contraction of the four intertwiners corresponding to the 1-cells converging at a vertex. More generally, for arbitrary *spin networks*, the vertex amplitude corresponds to $3nj$ -symbols, and $\langle s, s' \rangle_p$ takes the general form (151).

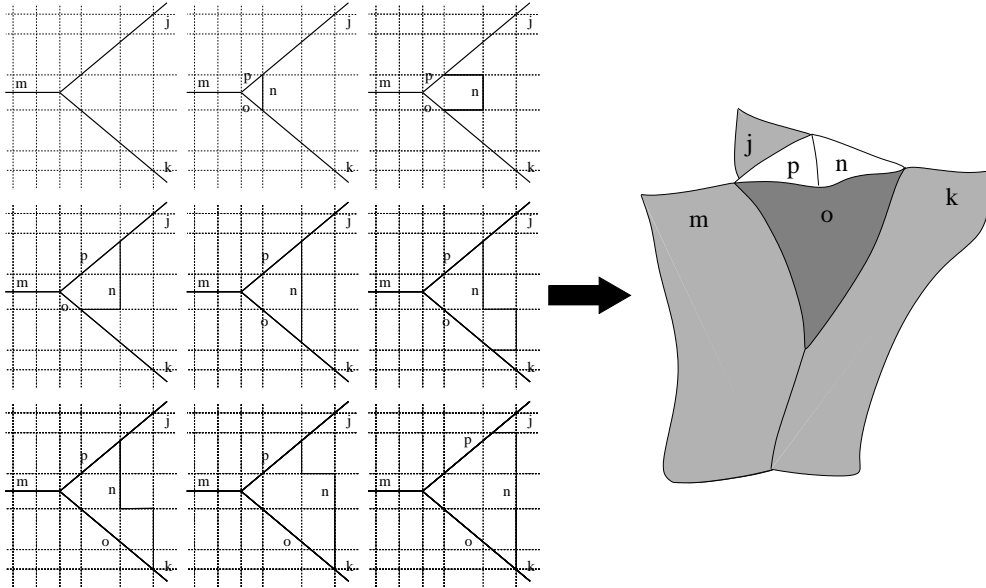


Fig. 16. A set of discrete transitions representing one of the contributing histories at a fixed value of the regulator. On the right, the continuous *spin foam* representation when the regulator is removed.

6.3. Spin foams from the covariant path integral

In this section we re-derive the SF-representation of the physical scalar product of 2+1 (Riemannian)^s quantum gravity directly as a regularization of the covariant

^sA generalization of the construction presented here for Lorentzian 2+1 gravity has been studied by Freidel⁹².

path integral^{93,94}. The formal path integral for 3d gravity can be written as

$$P = \int D[e]D[A] \exp \left[i \int_M \text{Tr}[e \wedge F(A)] \right]. \quad (169)$$

Assume $M = \Sigma \times I$, where $I \subset \mathbb{R}$ is a closed (time) interval (for simplicity we ignore boundary terms).

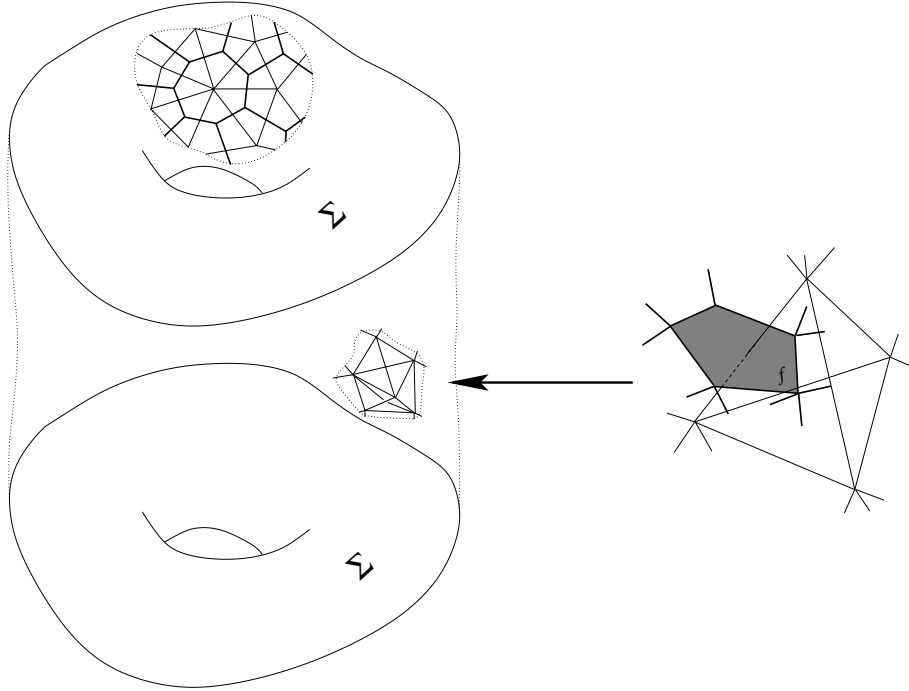


Fig. 17. The cellular decomposition of $M = \Sigma \times I$ ($\Sigma = T^2$ in this example). The illustration shows part of the induced graph on the boundary and the detail of a tetrahedron in Δ and a face $f \subset \Delta^*$ in the bulk.

In order to give a meaning to the formal expression above one replaces the 3-dimensional manifold (with boundary) M with an arbitrary cellular decomposition Δ . One also needs the notion of the associated dual 2-complex of Δ denoted by Δ^* . The dual 2-complex Δ^* is a combinatorial object defined by a set of vertices $v \subset \Delta^*$ (dual to 3-cells in Δ) edges $e \subset \Delta^*$ (dual to 2-cells in Δ) and faces $f \subset \Delta^*$ (dual to 1-cells in Δ). The intersection of the dual 2-complex Δ^* with the boundaries defines two graphs $\gamma_1, \gamma_2 \subset \Sigma$ (see Figure 17). For simplicity we ignore the boundaries until the end of this section. The fields e and A are discretized as follows. The $su(2)$ -valued 1-form field e is represented by the assignment of $e_f \in su(2)$ to each 1-cell in Δ . We use the fact that faces in Δ^* are in one-to-one correspondence with 1-cells in Δ and label e_f with a face subindex (Figure 17). The connection field A is

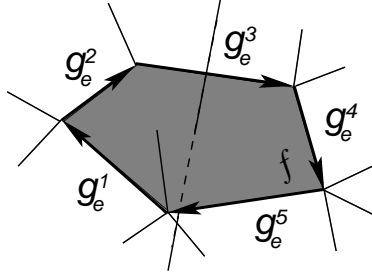


Fig. 18. A (2-cell) face $f \subset \Delta^*$ in a cellular decomposition of the spacetime manifold M . Also the corresponding dual 1-cell. The connection field is discretized by the assignment of the parallel transport group-elements $g_e^i \in SU(2)$ to edges $e \subset \Delta^*$ ($i = 1, \dots, 5$ in the face shown here).

represented by the assignment of group elements $g_e \in SU(2)$ to each edge in $e \subset \Delta^*$ (see Figure 18).

With all this (169) becomes the regularized version P_Δ defined as

$$P_\Delta = \int \prod_{f \subset \Delta^*} de_f \prod_{e \subset \Delta^*} dg_e \exp[i\text{Tr}[e_f W_f]], \quad (170)$$

where de_f is the regular Lebesgue measure on \mathbb{R}^3 , dg_e is the Haar measure on $SU(2)$, and W_f denotes the holonomy around (spacetime) faces, i.e., $W_f = g_e^1 \dots g_e^N$ for N being the number of edges bounding the corresponding face (see Figure 18). The discretization procedure is reminiscent of the one used in standard lattice gauge theory. The previous definition can be motivated by an analysis equivalent to the one presented in (160).

Integrating over e_f , and using (163), one obtains

$$P_\Delta = \sum_{\{j\}} \int \prod_{e \subset \Delta^*} dg_e \prod_{f \subset \Delta^*} (2j_f + 1) \text{Tr} \left[\prod_{j_f}^{j_f} (g_e^1 \dots g_e^N) \right]. \quad (171)$$

Now it remains to integrate over the lattice connection $\{g_e\}$. If an edge $e \subset \Delta^*$ bounds n faces $f \subset \Delta^*$ there will be n traces of the form $\text{Tr} \left[\prod_{j_f}^{j_f} (\dots g_e \dots) \right]$ in (171) containing g_e in the argument. In order to integrate over g_e we can use the following identity

$$I_{inv}^n := \int dg \prod_{j_1}^{j_1} (g) \otimes \prod_{j_2}^{j_2} (g) \otimes \dots \otimes \prod_{j_n}^{j_n} (g) = \sum_{\iota} C_{j_1 j_2 \dots j_n}^{\iota} C_{j_1 j_2 \dots j_n}^{*\iota}, \quad (172)$$

where I_{inv}^n is the projector from the tensor product of irreducible representations $\mathcal{H}_{j_1 \dots j_n} = j_1 \otimes j_2 \otimes \dots \otimes j_n$ onto the invariant component $\mathcal{H}_{j_1 \dots j_n}^0 = \text{Inv}[j_1 \otimes j_2 \otimes \dots \otimes j_n]$. On the r.h.s. we have chosen an orthonormal basis of invariant vectors (intertwiners) in $\mathcal{H}_{j_1 \dots j_n}$ to express the projector. Notice that the assignment of intertwiners to edges is a consequence of the integration over the connection. Using

the (172) one can write P_Δ in the general SF-representation form (151)

$$P_\Delta = \sum_{\{f\}} \prod_{f \in \Delta^*} (2j_f + 1) \prod_{v \in \Delta^*} A_v(j_v), \quad (173)$$

where $A_v(\iota_v, j_v)$ is given by the appropriate trace of the intertwiners corresponding to the edges bounded by the vertex. As in the previous section this amplitude is given in general by an $SU(2)$ $3Nj$ -symbol. When Δ is a simplicial complex all the edges in Δ^* are 3-valent and vertices are 4-valent. Consequently, the vertex amplitude is given by the contraction of the corresponding four 3-valent intertwiners, i.e., a $6j$ -symbol. In that case the path integral takes the (Ponzano-Regge⁹⁵) form

$$P_\Delta = \sum_{\{j\}} \prod_{f \in \Delta^*} (2j_f + 1) \prod_{v \in \Delta^*} \begin{array}{c} \text{---} j_4 \text{---} \\ \diagup \quad \diagdown \\ \text{---} j_3 \text{---} \\ \diagdown \quad \diagup \\ \text{---} j_6 \text{---} \\ \diagup \quad \diagdown \\ \text{---} j_1 \text{---} \quad \text{---} j_2 \text{---} \end{array}. \quad (174)$$

The labelling of faces that intersect the boundary naturally induces a labelling of the edges of the graphs γ_1 and γ_2 induced by the discretization. Thus, the boundary states are given by *spin network* states on γ_1 and γ_2 respectively. A careful analysis of the boundary contribution shows that only the face amplitude is modified to $(2j_f + 1)^{\nu_f/2}$, and that the *spin-foam* amplitudes are as in Equation (168).

A crucial property of the path integral in 3d gravity (and of the transition amplitudes in general) is that it does not depend on the discretization Δ —this is due to the absence of local degrees of freedom in 3d gravity and not expected to hold in 4d. Given two different cellular decompositions Δ and Δ' one has

$$\tau^{-n_0} P_\Delta = \tau^{-n'_0} P_{\Delta'}, \quad (175)$$

where n_0 is the number of 0-simplexes in Δ , and $\tau = \sum_j (2j + 1)^2$. This trivial scaling property of transition amplitudes allows for a simple definition of transition amplitudes that are independent of the discretization⁹⁶. However, notice that since τ is given by a divergent sum the discretization independence statement is formal. Moreover, the sum over spins in (174) is typically divergent. Divergences occur due to infinite gauge-volume factors in the path integral corresponding to the topological gauge freedom (154). Freidel and Louapre⁹⁷ have shown how these divergences can be avoided by gauge-fixing un-physical degrees of freedom in (170). In the case of 3d gravity with positive cosmological constant the state-sum generalizes to the Turaev-Viro invariant^{98,99,100,101} defined in terms of the quantum group $SU_q(2)$ with $q^n = 1$ where the representations are finitely many and thus $\tau < \infty$. Equation (175) is a rigorous statement in that case. No such infrared divergences appear in the canonical treatment of the previous section.

7. Spin foams in four dimensions

7.1. SF from the canonical formulation

There is no rigorous construction of the physical inner product of LQG in four dimensions. The *spin foam* representation as a device for its definition has been introduced formally by Rovelli¹⁰² and Rovelli and Reisenberger¹⁰³. In 4-dimensional LQG difficulties in understanding dynamics are centered around the quantum scalar constraint $\widehat{S} = \sqrt{\det E}^{-1} E_i^a E_j^b F_{ab}^{ij}(A) + \dots$ (see (63))—the vector constraint $\widehat{V}_a(A, E)$ is solved in a simple manner. The physical inner product formally becomes

$$\begin{aligned} \langle Ps, s' \rangle_{diff} &= \prod_x \delta[\widehat{S}(x)] = \int D[N] \langle \exp \left[i \int_{\Sigma} N(x) \widehat{S}(x) \right] s, s' \rangle_{diff} \\ &= \int D[N] \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle \left[\int_{\Sigma} N(x) \widehat{S}(x) \right]^n s, s' \rangle_{diff}, \quad (176) \end{aligned}$$

where \langle , \rangle_{diff} denotes the inner product in the Hilbert space of solutions of the vector constraint, and the exponential has been expanded in powers in the second line.

From early on, it was realized that smooth loop states are naturally annihilated by \widehat{S} (independently of any quantization ambiguity^{104,105}). Consequently, \widehat{S} acts only on *spin network* nodes. Generically, it does so by creating new links and nodes modifying the underlying graph of the *spin network* states (Figure 19).

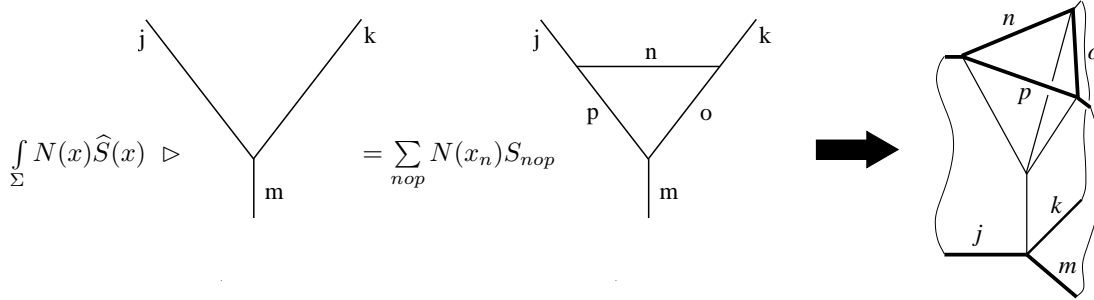


Fig. 19. The action of the scalar constraint and its *spin foam* representation. $N(x_n)$ is the value of N at the node and S_{nop} are the matrix elements of \widehat{S} .

Therefore, each term in the sum (176) represents a series of transitions—given by the local action of \widehat{S} at *spin network* nodes—through different *spin network* states interpolating the boundary states s and s' respectively. The action of \widehat{S} can be visualized as an ‘interaction vertex’ in the ‘time’ evolution of the node (Figure 19). As in the explicit 3d case, equation (176) can be expressed as sum over ‘histories’ of *spin networks* pictured as a system of branching surfaces described by a 2-complex whose elements inherit the representation labels on the intermediate

states. The value of the ‘transition’ amplitudes is controlled by the matrix elements of \widehat{S} . Therefore, although the qualitative picture is independent of quantization ambiguities transition amplitudes are sensitive to them.

Before even considering the issue of convergence of (176), the problem with this definition is evident: every single term in the sum is a divergent integral! Therefore, this way of presenting *spin foams* has to be considered as formal until a well defined regularization of (149) is provided. That is the goal of the *spin foam* approach.

Instead of dealing with an infinite number of constraints Thiemann recently proposed to impose one single *master* constraint defined as

$$M = \int_{\Sigma} dx^3 \frac{S^2(x) - q^{ab} V_a(x) V_b(x)}{\sqrt{\det q(x)}}. \quad (177)$$

Using techniques developed by Thiemann this constraint can indeed be promoted to a quantum operator acting on \mathcal{H}_{kin} . The physical inner product is given by

$$\langle s, s' \rangle_p := \lim_{T \rightarrow \infty} \langle s, \int_{-T}^T dt e^{it\widehat{M}} s' \rangle. \quad (178)$$

A SF-representation of the previous expression could now be achieved by the standard skeletonization that leads to the path integral representation in quantum mechanics. In this context one splits the t -parameter in discrete steps and writes

$$e^{it\widehat{M}} = \lim_{N \rightarrow \infty} [e^{it\widehat{M}/N}]^N = \lim_{N \rightarrow \infty} [1 + it\widehat{M}/N]^N. \quad (179)$$

The SF-representation follows from the fact that the action of the basic operator $1 + it\widehat{M}/N$ on a *spin network* can be written as a linear combination of new *spin networks* whose graphs and labels have been modified by the creation of new nodes (in a way qualitatively analogous to the local action shown in Figure 19). An explicit derivation of the physical inner product of 4d LQG along these lines is under current investigation.

7.2. Spin foams from the covariant formulation

In four dimensions the *spin foam* representation of the dynamics of LQG has been motivated by lattice discretizations of the path integral of gravity in the covariant formulation^{42,106,107,108}. This has lead to a series of constructions which are referred to as *spin foam models*⁸². These treatments are closer related to the construction of Section 6.3. Here we illustrate the formulation which has captured much interest in the literature: the *Barrett-Crane model* (BC model)^{109,110}.

7.2.1. Spin foam models for gravity as constrained quantum BF theory

The BC model is one of the most extensively studied *spin foam* models for quantum gravity. To introduce the main ideas involved we concentrate on the definition of

the model in the Riemannian sector. The BC model can be formally viewed as a *spin foam* quantization of $SO(4)$ Plebanski's formulation of GR¹¹¹. Plebanski's Riemannian action depends on an $so(4)$ connection A , a Lie-algebra-valued 2-form B and Lagrange multiplier fields λ and μ . Writing explicitly the Lie-algebra indices, the action is given by

$$I[B, A, \lambda, \mu] = \int [B^{IJ} \wedge F_{IJ}(A) + \lambda_{IJKL} B^{IJ} \wedge B^{KL} + \mu \epsilon^{IJKL} \lambda_{IJKL}], \quad (180)$$

where μ is a 4-form and $\lambda_{IJKL} = -\lambda_{JIKL} = -\lambda_{IJLK} = \lambda_{KLIJ}$ is a tensor in the internal space. Variation with respect to μ imposes the constraint $\epsilon^{IJKL} \lambda_{IJKL} = 0$ on λ_{IJKL} . The Lagrange multiplier tensor λ_{IJKL} has then 20 independent components. Variation with respect to λ imposes 20 algebraic equations on the 36 components of B . The (non-degenerate) solutions to the equations obtained by varying the multipliers λ and μ are

$$B^{IJ} = \pm \epsilon^{IJKL} e_K \wedge e_L, \quad \text{and} \quad B^{IJ} = \pm e^I \wedge e^J, \quad (181)$$

in terms of the 16 remaining degrees of freedom of the tetrad field e_a^I . If one substitutes the first solution into the original action one obtains Palatini's formulation of general relativity; therefore on shell (and on the right sector) the action is that of classical gravity.

The key idea in the definition of the model is that the path integral for the theory corresponding to the action $I[B, A, 0, 0]$, namely

$$P_{topo} = \int D[B]D[A] \exp \left[i \int [B^{IJ} \wedge F_{IJ}(A)] \right] \quad (182)$$

can be given a meaning as a *spin foam* sum, (151), in terms of a simple generalization of the construction of Section 6. In fact $I[B, A, 0, 0]$ corresponds to a simple theory known as BF theory that is formally very similar to 3d gravity¹¹². The result is independent of the chosen discretization because BF theory does not have local degrees of freedom (just as 3d gravity).

The BC model aims at providing a definition of the path integral of gravity pursuing a well-posed definition of the formal expression

$$P_{GR} = \int D[B]D[A] \delta [B \rightarrow \epsilon^{IJKL} e_K \wedge e_L] \exp \left[i \int [B^{IJ} \wedge F_{IJ}(A)] \right], \quad (183)$$

where $D[B]D[A] \delta (B \rightarrow \epsilon^{IJKL} e_K \wedge e_L)$ means that one must restrict the sum in (182) to those configurations of the topological theory satisfying the constraints $B = *(e \wedge e)$ for some tetrad e . The remarkable fact is that this restriction can be implemented in a systematic way directly on the *spin foam* configurations that define P_{topo} ^{113,114}.

In P_{topo} *spin foams* are labelled with spins corresponding to the unitary irreducible representations of $SO(4)$ (given by two spin quantum numbers (j_R, j_L)). Essentially, the factor ' $\delta (B \rightarrow \epsilon^{IJKL} e_K \wedge e_L)$ ' restricts the set of *spin foam* quantum numbers to the so-called simple representations (for which $j_R = j_L = j$ ^{87,115}).

This is the ‘quantum’ version of the solution to the constraints (181). There are various versions of this model, some versions satisfy intriguing finiteness^{116,117,118,119} properties^t. The simplest definition of the transition amplitudes in the BC -model is given by

$$P(s^*s) = \sum_{\{j\}} \prod_{f \subset F_{s \rightarrow s'}} (2j_f + 1)^{\nu_f} \prod_{v \subset F_{s \rightarrow s'}} \sum_{\ell_1 \dots \ell_5} \left(\begin{array}{c} \ell_2 \\ j_{12} \quad j_{23} \\ j_{13} \\ j_{25} \quad j_{24} \\ j_{14} \quad j_{35} \\ j_{34} \\ \ell_3 \\ j_{15} \quad j_{45} \quad \ell_4 \\ \ell_5 \end{array} \right) \left(\begin{array}{c} \ell_2^* \\ j_{12}^* \quad j_{23}^* \\ j_{13}^* \\ j_{25}^* \quad j_{24}^* \\ j_{14}^* \quad j_{35}^* \\ j_{34}^* \\ \ell_3^* \\ j_{15}^* \quad j_{45}^* \quad \ell_4^* \\ \ell_5^* \end{array} \right), \tag{184}$$

where we use the notation of (168), the graphs denote $15j$ symbols, and ℓ_i are half integers labelling $SU(2)$ normalized 4-intertwiners^u. No rigorous connection with the Hilbert space picture of LQG has yet been established. The self-dual version of Plebanski’s action leads, through a similar construction, to Reisenberger’s model¹⁰⁷. A general prescription for the definition of constrained BF theories^v on the lattice has been studied by Freidel and Krasnov¹²³. Lorentzian generalizations of the Barrett-Crane model have been defined^{124,125}. A generalization using quantum groups was studied by Noui and Roche¹²⁶.

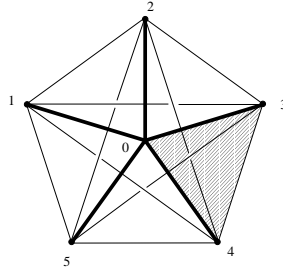


Fig. 20. The dual of a 4-simplex.

The simplest amplitude in the BC model corresponds to a single 4-simplex. A 4-simplex can be viewed as the simplest triangulation of the 4-dimensional space time given by the interior of a 3-sphere (the corresponding 2-complex is shown in Figure 20). States of the 4-simplex are labelled by 10 spins j (labelling the 10 edges of the boundary *spin network*, see Figure 20) which can be shown to be related to the area in Planck units of the 10 triangular faces that form the 4-simplex. A first indication of the connection of the model with gravity was that

^tFor a discussion of the freedom involved see¹²⁰.
^uReisenberger¹²¹ proved that the 4-simplex BC amplitude is unique up to normalization.
^vGambini and Pullin studied an alternative modification of BF theory leading to a simple model with intriguing properties¹²².

the large j -asymptotics appeared to be dominated by the exponential of the Regge action^{127,128} (the action derived by Regge as a discretization of GR). This estimate was done using the stationary phase approximation to the integral¹²⁹ that gives the amplitude of a 4-simplex in the BC model. However, more detailed calculations showed that the amplitude is dominated by configurations corresponding to degenerate 4-simplexes^{130,97,131}. This seem to invalidate a simple connection to GR and is one of the main puzzles in the model.

7.3. Spin foams as Feynman diagrams

The main problem with the models of the previous section is that they are defined on a discretization Δ of M and that—contrary to what happens with a topological theory, e.g. 3d gravity (Equation 175)—the amplitudes depend on the discretization Δ . Various possibilities to eliminate this regulator have been discussed in the literature but no explicit results are yet known in 4d. An interesting proposal is a discretization-independent definition of *spin foam* models achieved by the introduction of an auxiliary field theory living on an abstract group manifold— $Spin(4)^4$ and $SL(2, C)^4$ for Riemannian and Lorentzian gravity respectively^{132,133,134,135,136}. The action of the auxiliary group field theory (GFT) takes the form

$$I[\phi] = \int_{G^4} \phi^2 + \frac{\lambda}{5!} \int_{G^{10}} M^{(5)}[\phi], \quad (185)$$

where $M^{(5)}[\phi]$ is a fifth order monomial, and G is the corresponding group. In the simplest model $M^{(5)}[\phi] =$

$$\phi(g_1, g_2, g_3, g_4)\phi(g_4, g_5, g_6, g_7)\phi(g_7, g_3, g_8, g_9)\phi(g_9, g_6, g_2, g_{10})\phi(g_{10}, g_8, g_5, g_1).$$

The field ϕ is required to be invariant under the (simultaneous) right action of the group on its four arguments in addition to other symmetries (not described here for simplicity). The perturbative expansion in λ of the GFT *Euclidean* path integral is given by

$$P = \int D[\phi] e^{-I[\phi]} = \sum_{F_N} \frac{\lambda^N}{\text{sym}[F_N]} A[F_N], \quad (186)$$

where $A[F_N]$ corresponds to a sum of Feynman-diagram amplitudes for diagrams with N interaction vertices, and $\text{sym}[F_N]$ denotes the standard symmetry factor. A remarkable property of this expansion is that $A[F_N]$ can be expressed as a sum over spin foam amplitudes, i.e., 2-complexes labelled by unitary irreducible representations of G . Moreover, for very simple interaction $M^{(5)}[\phi]$, the *spin foam* amplitudes are in one-to-one correspondence to those found in the models of the previous section (e.g. the BC model). This *duality* is regarded as a way of providing a fully combinatorial definition of quantum gravity where no reference to any discretization or even a manifold-structure is made. Transition amplitudes between *spin network* states correspond to n -point functions of the field theory¹³⁷. These models have been inspired by generalizations of matrix models applied to BF theory^{138,139}.

Divergent transition amplitudes can arise by the contribution of ‘loop’ diagrams as in standard QFT. In *spin foams*, diagrams corresponding to 2-dimensional bubbles are potentially divergent because spin labels can be arbitrarily high leading to unbounded sums in (151). Such divergences do not occur in certain field theories dual (in the sense above) to the Barrett-Crane (BC) model. However, little is known about the convergence of the series in λ and the physical meaning of this constant. Nevertheless, Freidel and Louapre¹⁴⁰ have shown that the series can be re-summed in certain models dual to lower dimensional theories. Techniques for studying the continuum limit of these kind of theories have been proposed^{141,142,143,144}. There are models defined in this context admitting matter degree of freedom^{145,146}.

7.4. Causal spin foams

Let us finish by presenting a fundamentally different construction leading to *spin foams*. Using the kinematical setting of LQG with the assumption of the existence of a micro-local (in the sense of Planck scale) causal structure Markopoulou and Smolin^{147,148,149} define a general class of (causal) *spin foam* models for gravity. The elementary transition amplitude $A_{s_I \rightarrow s_{I+1}}$ from an initial *spin network* s_I to another *spin network* s_{I+1} is defined by a set of simple combinatorial rules based on a definition of causal propagation of the information at nodes. The rules and amplitudes have to satisfy certain causal restrictions (motivated by the standard concepts in classical Lorentzian physics). These rules generate surface-like excitations of the same kind one encounters in the previous formulations. *Spin foams* $F_{s_i \rightarrow s_f}^N$ are labelled by the number of times, N , these elementary transitions take place. Transition amplitudes are defined as

$$\langle s_i, s_f \rangle = \sum_N A(F_{s_i \rightarrow s_f}^N) \quad (187)$$

which is of the generic form (151). The models are not related to any continuum action. The only guiding principles in the construction are the restrictions imposed by causality, and the requirement of the existence of a non-trivial critical behavior that reproduces general relativity at large scales. Some indirect evidence of a possible non trivial continuum limit has been obtained in certain versions of these models in $1+1$ dimensions.

8. Some final bibliographic remarks

We did not have time to discuss the applications of loop quantum gravity to cosmology. The interested reader is referred to review article¹⁵⁰ and the references therein.

We did not have the chance to mention in this lectures the important area of research in LQG devoted to the study of the low energy limit of the theory. We refer the reader to the general sources^{19,4,20} for bibliography and an overview of results and outlook.

In Section 4 we introduced a representation of the basic kinematical observables in the kinematical Hilbert space \mathcal{H}_{kin} . That was the starting point for the definition of the theory. The reader might wonder why one emphasizes so much the Hilbert space as a fundamental object when in the context of standard quantum field theory it is rather the algebra of observables what plays the fundamental role. Hilbert spaces correspond to representations of the algebra of observables which are chosen according to the physical situation at hand. However, when extra symmetry is present it can happen that there is no freedom and that a single representation is selected by the additional symmetry. This is in fact the case in LQG if one imposes the condition of diffeomorphism invariance on the state that defines the representation of the holonomy and the flux operators 151,152,153,154,155,156,157.

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