TAMING THE CONFORMAL ZOO

Gregory MOORE and Nathan SEIBERG [1]
Institute for Advanced Study, Princeton, NJ 08540, USA

Received 18 January 1989

All known rational conformal field theories may be obtained from (2+1)-dimensional Chern–Simons gauge theories by appropriate choice of gauge group. We conjecture that all rational field theories are classified by groups via (2+1)-dimensional Chern–Simons gauge theories.

1. Introduction

The problem of the classification of all conformal field theories is a useful problem to orient the research about the more interesting and more important problem of uncovering the meaning of conformal field theory, and, perhaps, string theory. An interesting subset of all conformal field theories are the rational conformal field theories (RCFT). These appear to be more tractable than the more general theories because they have only a *finite* number of primary fields. In refs. [1,2] it was noticed that the modular invariants of a chiral algebra $\mathcal{A}$ are either diagonal or related to the diagonal by an automorphism of the fusion rule algebra. This reduces the problem of the classification of all RCFT's to the classification of all chiral algebras. Chiral algebras themselves may be studied by characterizing their representations in terms of polynomial equations [3] which are similar to those defining groups. Indeed it has been widely recognized (see, e.g., refs. [3–5]) that quantum groups can be used to generate solutions of these equations.

A somewhat different perspective on the classification problem has been provided by the remarkable observation of Witten [6] that current algebra in two dimensions is equivalent to Chern–Simons gauge theory (CSGT) in three dimensions and that Friedan-Shenker modular geometry [7] and some of the structure uncovered in ref. [3] is neatly summarized by 3D general covariance. In ref. [6] the connection between two- and three-dimensional theories was established only for WZW models [8] based on a simply connected compact Lie group G. In this letter we show that all known RCFT's are equivalent to some CSGT thus organizing the entire zoo of known RCFT's simply by a choice of gauge group and coupling constants.

The relation between the 3D and the 2D theories arises in two (related) ways. First, the states in the Hilbert space of the CSGT on a compact surface is the space of conformal blocks of the two-dimensional RCFT [6]. Second, when quantizing the system on a manifold with a boundary, one can recover all the states of the chiral algebra and its representations. The second viewpoint will be particularly useful in our discussion so we begin by rephrasing [9] the discussion at the end of ref. [6] (based on phase spaces and symplectic forms) in lagrangian terms.

Consider the path integral of the CSGT on a three-manifold Y. Boundary conditions on the gauge fields A can be determined by requiring that there are no boundary corrections to the equations of motion. Since the variation of the Chern–Simons action

$$S=k\,\mathrm{CS}(A)\equiv\frac{k}{4\pi}\int_Y \mathrm{Tr}(A\mathrm{d}A+\tfrac{2}{3}A^3) \tag{1.1}$$

(where we use the normalization $\mathrm{Tr}\,T^aT^b=-\frac{1}{2}\delta^{ab}$) is given by

[1] On leave of absence from the Department of Physics, Weizmann Institute of Science, Rehovot 76100, Israel.

$$\delta S=\frac{k}{4\pi}\int_{\delta Y}\mathrm{Tr}(\delta A A)+\frac{k}{2\pi}\int_{Y}\mathrm{Tr}(\delta A F)$$

we must break general covariance on the boundary and set one of the components of A (say A_0) to zero [#1]. With these boundary conditions the functional integral is invariant only under gauge transformations which are one at the boundary.

Having specified the boundary conditions we can evaluate the functional integral. If Y has a "space-time splitting" of the form $\Sigma\times R$ where Σ is a Riemann surface (perhaps with boundary), decompose the exterior derivative $d=dt\,\partial/\partial t+\tilde{d}$ and the gauge field $A=A_t+\tilde{A}$ into time and space components. The CS action becomes

$$S=-\frac{k}{4\pi}\int_{Y}\mathrm{Tr}\left(\tilde{A}\frac{\partial}{\partial t}\tilde{A}\,dt\right)+\frac{k}{2\pi}\int_{Y}\mathrm{Tr}(A_t(\tilde{d}\tilde{A}+\tilde{A}^2))\,.$$

The functional integral over A_t imposes the constraint $F=0$ in the space directions.

Now let $\Sigma=D$ be a disk. The delta function $\delta(F)$ is easily solved to give $\tilde{A}=-\tilde{d}U\,U^{-1}$ for a single-valued map $U: D\times R\to G$. Plugging back into the action and integrating by parts we find (φ is the angular coordinate on ∂D)

$$S=k\,S_{WZW}(U)\equiv\frac{k}{4\pi}\int_{\partial Y}\mathrm{Tr}(U^{-1}\partial_\varphi U\,U^{-1}\partial_t U)\,d\varphi\,dt+\frac{k}{12\pi}\int_{Y}\mathrm{Tr}(U^{-1}\tilde{d}U)^3\,, \qquad (1.2)$$

which depends only on the boundary values of U. Moreover, from this change of variables we obtain the Haar measure $DA\,\delta(F)=DU$ so, factoring out the volume of the gauge group we recover the chiral version of the WZW path integral. This lagrangian is invariant under the transformation on the boundary $U(\varphi,t)\to\tilde{V}(\varphi)UV(t)$. The variance under $\tilde{V}$ is a global symmetry because it does not go to one in the past and in the future. The gauge symmetry $V(t)$ reflects a redundancy in our parametrization of A by U and has to be fixed. Since the lagrangian is first order in time derivatives we recover the phase space as the space of based loops LG/G together with the symplectic structure [8]

$$\omega_0=\frac{k}{4\pi}\oint\mathrm{Tr}(g^{-1}\delta g)\frac{d}{d\sigma}(g^{-1}\delta g)\,.$$

Quantization of this system gives the basic representation of LG [8,10] from which one obtains the chiral algebra of the G current algebra. In particular the boundary values of the gauge field A_φ become operators satisfying the commutation relations of Kač–Moody currents.

When Σ is more complicated U will not in general be single-valued. For example, on the annulus, giving U an appropriate holonomy under $\varphi\to\varphi+2\pi$ we reproduce the symplectic structure

$$\omega=\omega_0+\oint\mathrm{Tr}\,t_0(g^{-1}\delta g)^2$$

on the phase space associated to each boundary, where t_0 defines a coadjoint orbit of (a central extension of) LG. Quantization of this system gives the other integrable representations of LG [10].

Finally, consider the effect of shrinking a boundary to a point. As a hole shrinks to a point a gauge field with holonomy becomes a gauge transform of a smooth gauge field by a singular gauge transformation. Consider a closed loop in C in $Y=R^2\times S^1$ winding around the S^1 direction and not linking any Wilson loop. If we perform a gauge transformation which, near C, is of the form $g(\phi)=\exp(\alpha\phi)$, for some element α of the Lie algebra (ϕ is an angular variable), then the value of the path integral will change. Therefore, performing a singular gauge transformation along a curve defines an operator [#2]. In CSGT such operators are equivalent to the Wilson operators, as we now demonstrate. Regularize the operator by first cutting out a small tube around C

[#1] To specify a non-zero value for A_0 at the boundary, add a term $(1/4\pi)\int_{\partial Y}\mathrm{Tr}A_0A_1$ to the action (A_1 is the component of A along a direction not parallel to A_0). This boundary term can also be derived by demanding that the functional integral description of the quantum system coincides with the canonical formalism [9].

[#2] If other operators link C, this simple construction works only for $\exp(\alpha)$ in the center of G. The operator constructed this way is known as 't Hooft operator [11].

and then performing the gauge transformation $g(\phi)$. The action changes by [#3]

$$S \to S - \frac{k}{2\pi} \int_{\partial Y} \mathrm{Tr}\, A g^{-1} \mathrm{d}g \, .$$

To define a gauge-invariant operator, average over gauge transformations $h(t)g(\phi)h^{-1}(t)$. Physically, averaging quantizes the collective coordinates of the soliton created by the operator. Thus, the effect of this loop operator on the path integral is

$$\int \mathrm{D}A \exp(\mathrm{i}S) \to \int \mathrm{D}A \, I_{\mathrm{C}}(A) \exp(\mathrm{i}S) \, ,$$

where

$$I_{\mathrm{C}}(A) = \int \mathrm{D}h(t) \exp\left(-\mathrm{i}\frac{k}{2} \oint_{\mathrm{C}} \mathrm{Tr}(\alpha h(\mathrm{d}+A)h^{-1}) \right).$$

The action depends only on the degrees of freedom describing the (coadjoint) orbit $h^{-1}(t)\alpha h(t)$. Thus, separating the degrees of freedom we find that $I_{\mathrm{C}}(A)$ is, up to a constant, simply the path integral of a well-known quantum mechanics problem describing the quantization of the orbit of $\frac{1}{2}k\alpha$ with the hamiltonian given by the function $\frac{1}{2}k \mathrm{Tr}\, A^{-1}(t)\alpha h(t)$ on the orbit (see, e.g. ref. [12]). In particular the path integral only makes sense if $\frac{1}{2}k\alpha = \lambda$ is a weight of the group, in which case we can replace the path integral by

$$I_{\mathrm{C}}(A) \to W_{\lambda}(\mathrm{C}) = \mathrm{Tr}_{\lambda}\, \mathrm{P} \exp\left(\mathrm{i} \oint_{\mathrm{C}} A \right),$$

thus completing the demonstration of the equivalence with Wilson loops.

2. Extensions of affine algebras

The CSGT for non-simply connected compact Lie groups $\mathrm{G} = \tilde{\mathrm{G}}/\mathrm{Z}$ (for $\tilde{\mathrm{G}}$ simply connected with center Z) exhibits new features [#4]. The actions for the G and $\tilde{\mathrm{G}}$ theories are the same, but the path integrals differ in the integration region, since there are G bundles which do not lift to $\tilde{\mathrm{G}}$ bundles. Normalizing the roots α of $\tilde{\mathrm{G}}$ so that $\exp(2\pi\alpha) = 1$ (and hence $\exp(2\pi\mu)$ is central for weights μ), topologically nontrivial G-bundles on Riemann surfaces are constructed from transition functions of the form $g_{\mu}(\phi) \equiv \exp(\phi\mu)$ where ϕ is some angular coordinate. In the $\tilde{\mathrm{G}}$ theory the action of an 't Hooft operator associated to g_{μ} parallel to some Wilson line $W_{\lambda}(\mathrm{C})$ (ϕ winds around C), produces a new Wilson line $W_{\mu(\lambda)}(\mathrm{C})$ associated to the holonomy $\exp[2\pi(\mu+\lambda/k)]$. The mapping $\lambda \to \mu(\lambda)$ of current algebra representations is well-known in conformal field theory as the spectral flow operation associated to the central element $\exp(2\pi\mu)$ (which in turn is usually associated with a corresponding automorphism of the extended Dynkin diagram) [13–15]. The flow operation is simply characterized as the mapping of representations induced by the transformation

$$J(z) \to \Omega(z) J(z) \Omega^{-1}(z) - k\partial\Omega(z)\Omega^{-1}(z)$$

of the currents [#5] where $\Omega(z) \equiv z^{\mu}$. Thus in the G theory the Wilson lines in representations related by spectral flow are equivalent. Correspondingly in 2D CFT there is an extended chiral algebra, extending the ordinary $\tilde{\mathrm{G}}$ current algebra, $\mathscr{A}(\tilde{\mathrm{G}})$. Denoting by $\mathscr{H}_{\lambda}$ the integrable representations of the $\tilde{\mathrm{G}}$ current algebra we have the chiral algebra $\mathscr{A}(\mathrm{G}) = \oplus_{\mu} \mathscr{H}_{\mu(0)}$ (where the sum over μ is over the elements of Z) and similarly, the representations of $\mathscr{A}(\tilde{\mathrm{G}})$ are $\mathscr{H}_{[\lambda]} = \oplus_{\mu} \mathscr{H}_{\mu(\lambda)}$.

Path integrals in the G theory can be expressed as path integrals in the $\tilde{\mathrm{G}}$ theory. Since the 't Hooft loop associated to g_{μ} is trivial in the G path integral and creates the Wilson line $W_{(k/2)\mu}$ in the $\tilde{\mathrm{G}}$ path integral we can think of the G theory as a $\tilde{\mathrm{G}}$ theory in which the 't Hooft solitons have condensed. The G theory functional integral is a sum of functional integrals in the $\tilde{\mathrm{G}}$ theory with insertions of $\Pi(\mathrm{C}) = \Sigma_{(k/2)\mu}(\mathrm{C})$ for every curve C which is a generator of $\mathrm{H}^{1}(\mathrm{Y})$. For instance, for the three-manifold $\mathrm{S}^2 \times \mathrm{S}^1$ we can evaluate the expectation values of Wilson lines on C_a as

[#3] In deriving this we have to be careful about the boundary terms discussed above, accounting for the "extra" factor of 2.

[#4] The following discussion is easily modified if we use only a subgroup of the center.

[#5] Recall that since the tangential gauge field A_{φ} becomes the current in two dimensions this is the 2D analog of the 't Hooft operator.

$$\int_{\mathrm{G\ theory}} \mathrm{D}A \prod_{a=1}^{n} W_{[\lambda_a]}(\mathrm{C}_a)\exp(\mathrm{i}S)$$

$$= \int_{\tilde{\mathrm{G}}\ \mathrm{theory}} \mathrm{D}A\, \Pi(\mathrm{C}_t) \prod_{a=1}^{n} W_{\lambda_a}(\mathrm{C}_a)\exp(\mathrm{i}S) \quad (2.1)$$

where C_t winds around the S^1 direction. In this expression we should include only Wilson lines which are invariant under G gauge transformations. This amounts to insertions of the projection operator $\Pi(\mathrm{C}_{\tilde{a}})/|\mathrm{Z}|$ where $\mathrm{C}_{\tilde{a}}$ winds around C_a and $|\mathrm{Z}|$ is the order of Z. We could also insert this projection operator with $\mathrm{C}_{\tilde{a}}$ cabling (i.e. parallel nonbraiding relative to) C_a. Cabled Wilson lines in representations i, j satisfy the fusion rule algebra [16] of the corresponding CFT [6]:

$$W_i(\mathrm{C})\mathrm{W}_j(\mathrm{C}) = \sum_k N_{ij}^k W_k(\mathrm{C}) .$$

Hence, we can rewrite the above path integrals as

$$|\mathrm{Z}| \int_{\tilde{\mathrm{G}}\ \mathrm{theory}} \mathrm{D}A$$

$$\times \prod_{a=1}^{n} \left(\frac{1}{|\mathrm{Z}|} \sum_{\mu\in\mathrm{Z}} W_{\mu(\lambda_a)}(\mathrm{C}_a) \right) \exp(\mathrm{i}S) .$$

From these considerations we can draw several conclusions. First, unlike $\mathscr{A}(\tilde{\mathrm{G}})$, the chiral algebra $\mathscr{A}(\mathrm{G})$ does not makes sense for every integer value of k. From the point of view of three dimensions the Wilson lines $W_{(k/2)\mu}$ should be invisible in the G path integral, in particular they should not depend on framing [6]. From the point of view of two dimensions the fields in the extending representations $\mathscr{H}_{\mu(0)}$ must be mutually local, in particular they should have integer weights. In any case, the restriction on k is that $\langle k\mu, k\mu+2\rho\rangle = 0 \bmod 2(k+h)$ where h is the dual coxeter number and ρ half the sum of positive roots of $\tilde{\mathrm{G}}$. Not surprisingly, $\mathscr{A}(\mathrm{G})$ is the chiral algebra of the WZW theory based on the group G [17,14,15] for these values of k. Precisely for these values the WZW theory has a diagonal partition function.

To enumerate the observables of the G theory we must keep in mind the following three rules:

Rule (a): Since the G theory should not be sensitive to the precise location of the insertions of $W_{(k/2)\mu}$, the allowed Wilson operators W_λ should have trivial braiding properties with respect to $W_{(k/2)\mu}$. Equivalently, we require the Wilson lines to be gauge invariant, even under the gauge transformations $g_\mu(\phi)$ (with ϕ running along the loop). Therefore λ is also a weight of G. From the 2D point of view we require primary fields to be mutually local with respect to the extending fields $\mathscr{H}_{\mu(0)}$.

Rule (b): The operators W_λ and $W_{\mu(\lambda)}$ are equivalent. In 2D language the representations $\mathscr{H}_{[\lambda]}$ of $\mathscr{A}(\mathrm{G})$ have the form $\mathscr{H}_{[\lambda]} = \oplus_\mu \mathscr{H}_{\mu(\lambda)}$. There are corresponding relations between the spaces of conformal blocks and the braiding matrices for $\mathscr{A}(\mathrm{G})$ and $\mathscr{A}(\tilde{\mathrm{G}})$. For example, considering conformal blocks as invariant tensors determined by Ward identities we have

$$\mathrm{Inv}(\mathscr{H}_{[\lambda_1]}, ..., \mathscr{H}_{[\lambda_m]})_{\mathrm{G}} = \bigoplus_\mu \mathrm{Inv}(\mathscr{H}_{\mu(\lambda_1)}, \mathscr{H}_{\lambda_2}, \mathscr{H}_{\lambda_m})_{\tilde{\mathrm{G}}} .$$

Rule (c): $\mathscr{H}_{[\lambda]}$ is reducible if λ is fixed by some spectral flow $\mu(\lambda) = \lambda$. In 2D CFT $\mathscr{H}_{[\lambda]}$ is a direct sum of equivalent $\mathscr{A}(\tilde{\mathrm{G}})$ representations. The operators which extend $\mathscr{A}(\tilde{\mathrm{G}})$ act with phases which distinguish these as distinct $\mathscr{A}(\mathrm{G})$ representations. In 3D CSGT there are new operators which cannot be represented by Wilson lines but rather must be represented by a Wilson line with a line emanating perpendicularly, where the emanating line carries one of the extending representations $(k/2)\mu$.

The above picture can be checked by quantizing the G-theory on special Riemann surfaces. Consider first the quantization on the disk. Repeating the discussion from (1.1) to (1.2), we again find the phase space LG/G. Since $\pi_0(\mathrm{LG/G}) = \pi_1(\mathrm{G})$, the phase space and Hilbert space are decomposed into sectors labeled by the elements of $\pi_1(\mathrm{G})$. (Since the gauge group of the theory forces $g=1$ at the boundary there are non-trivial G bundles on the disk which are classified by $\pi_1(\mathrm{G})$.) Quantizing in the nontrivial sectors amounts to quantizing fields U which, when lifted to $\tilde{\mathrm{G}}$ have holonomy $\exp(2\pi\mu)$. By the above discussion of 't Hooft loops we recover the previous description of the chiral algebra. This description can also be recovered from the Borel–Weil theorem [15].

Proceeding now to the torus we must quantize the phase space given by the space of flat G bundles. This phase space decomposes into several components.

The space of flat G bundles which lift to $\tilde{G}$ bundles is $T\times T/W$ where T is the maximal torus of G and the Weyl group W acts diagonally. Since T is $\tilde{T}/Z$, and since the $\tilde{G}$ physical states are the Weyl–Kač characters, the characters which descent from $\tilde{T}\times\tilde{T}/W$ to $T\times T/W$ are in one-to-one correspondence with the space $\mathscr{H}_{[\lambda]}$ mentioned in rules (a), (b) above. Note that in the semi-classical limit the number of physical states has been reduced by a factor of $|Z|^2$ (one factor of $|Z|$ corresponds to rule (a) and another one to rule (b)), as expected since the volume of the classical phase space is reduced by $|Z|^2$. The remaining components of phase space, parametrizing G bundles which do not lift to $\tilde{G}$ bundles are simply points corresponding to conjugacy classes of solutions $(A, B)\tilde{\in}G\times\tilde{G}$ to $ABA^{-1}B^{-1}=\exp(2\pi\mu)$ [18]. These solutions correspond to the solutions of the fixed-point condition $\mu(\lambda)=\lambda$. Each point in phase space contributes one quantum state, thus reproducing the counting of representations implied by rule (c). The states associated to twisted G bundles on the torus correspond to the conformal blocks for the torus one point functions of $\tilde{G}$ current algebra for the representations $\mu(0)$.

Two simple examples illustrate these ideas. First consider $U(1)=R/\mathbb{Z}$. The special representations μ have charges $(k/2)\mathbb{Z}$. Since $\Delta_\mu=\frac{1}{2}(k/2)^2$ has to be an integer, k must be divisible by 4. Rule (a) excludes Wilson loops with fractional charges. Rule (b) states that loops with charges n and $n+k/2$ have the same correlation functions. Since the spectral flow has no fixed point, rule (c) does not apply (there are no non-trivial flat bundles). We conclude that this theory leads to the chiral algebra of the rational torus with level $N=k/4$. The 't Hooft operator, $W_{k/2}$, is the 3D version of the operator $\exp(i\sqrt{2N}X)$ which extends the Kač–Moody algebra in 2D theory. Notice that $\mathscr{A}(R)\subset\mathscr{A}(U(1))$. $\mathscr{A}(R)$ is generated by ∂X and is usually referred to as (the enveloping algebra of) U(1) Kač–Moody. $\mathscr{A}(U(1))$, which depends on the level will also be denoted by $U(1)_N$. As a second example take $SO(3)=SU(2)/\mathbb{Z}_2$. In this case, the special representation μ has spin $k/2$. Since $\Delta_\mu=(k/2)(k/2+1)/(k+2)$ has to be an integer, k has to divisible by 4. Rule (a) excludes Wilson loops with half integral spins. Rule (b) guarantees that loops with spin j and $k/2-j$ have the same correlation functions. The fixed point of the spectral flow, $j=k/4$, corresponds to two different representations of the chiral algebra (rule (c)). We conclude that this theory leads to the chiral algebra obtained in SO(3) WZW theories at level $k=0 \bmod 4$ [17,1,2,19]. Notice that $\mathscr{A}(SU(2))\subset\mathscr{A}(SO(3))$.

3. Coset constructions

The previous construction can obviously be extended to include products of gauge groups and various divisions by subgroups of the centers. A more interesting situation arises if one considers a tensor product of gauge groups $G_1\times\ldots\times G_n$ where the couplings k_i of the gauge groups are allowed to have both positive and negative signs. Since the overall sign of the action can be changed by reversing the orientation of the manifold, only the relative signs of k_i are significant. Consider for example the theory based on the gauge group $G\times H$ with the action $k_G CS(A)-k_H CS(B)$ where A is a G gauge field and B is an H gauge field. The Wilson line operators are $\mathrm{tr}_\Lambda(P\exp(\oint A))\,\mathrm{tr}_\lambda(P\exp(\oint B))$. Since we can change the sign of k_H by the transformation $B(x)\to -B(-x)$, quantization on compact Riemann surfaces leads to quantum states which are the conformal blocks of a holomorphic G theory times the conformal blocks of an antiholomorphic H theory. Correspondingly, the duality matrices are those of G times the inverse of those of H. When quantizing on the disk with the boundary conditions discussed in section one we find the algebra $\mathscr{A}(G)\times\overline{\mathscr{A}(H)}$.

If $H\subset G$ and $k_G=lk_H$, where l is the index of the embedding, there is another possible boundary condition [#6]. Let $Y=D\times R$ and recall that the boundary conditions have to be consistent with the equations of motion, and therefore $k_G\mathrm{Tr}\,A\delta A-k_H\mathrm{Tr}\,B\delta B=0$. Let π_H be the orthogonal projection in the Killing metric of the Lie algebra of G onto the Lie algebra of H. We may then impose $\pi_H(A)=B$ for the tangential directions of the gauge fields and $A_0-\pi_H(A_0)=0$ for the time direction. These boundary conditions leave a gauge group of maps $(g, h): D\to G\times H$ which reduce to the diagonal map (h, h) on the boundary. In fact the boundary condition allows a holonomy

[#6] For convenience we take $l=1$ henceforth.

$\exp(2\pi\mu)$ for h. Thus the gauge group of the CSGT is $(G\times H)/Z$ where Z is the *common* center of G and H. Following manipulations analogous to those used in passing from (1.1) to (1.2) the path integral on $D\times R$ becomes, after factoring out the gauge group volume, an integral over the boundary values of fields:

$$\int D\lambda\, DU\, DV \exp\left(ikS_{WZW}(U) - ikS_{WZW}(V) + ik\int \mathrm{Tr}\,\lambda(\partial_\varphi U\, U^{-1} - \partial_\varphi V\, V^{-1})\right), \qquad (3.1)$$

where the path integral on λ in the Lie algebra of H enforces the boundary condition. If we define $U=gV$, $a_t=\lambda$, and $a_\varphi=-\partial_\varphi V\, V^{-1}$ then a simple application of the Polyakov–Wiegmann formula shows that (3.1) is the path integral of the gauged WZW model [#7]. It is known [20] that the gauged WZW action reproduces the coset G/H theory. We may understand this since quantization of (3.1) where the fields U, V have holonomies appropriate to representations Λ, λ of LG, LH, respectively, leads to the space of quantum states $\mathscr{H}_\Lambda \otimes \mathscr{H}^*_\lambda$. The difference of the H currents $\pi_H(\partial_\varphi U\, U^{-1}) - \partial_\varphi V\, V^{-1}$ is an LH affine Lie algebra with zero central extension, so the vanishing of these operators on the physical states is a set of first class constraints. Imposing these constraints restricts the physical states to the LH invariant states in the tensor product, which is simply the space of LH intertwiners between $\mathscr{H}_\lambda$ and the representation $\mathscr{H}_\Lambda$, considered as an LH representation. In formulas:

$$\mathscr{H}_{\Lambda,\lambda} \equiv (\mathscr{H}_\Lambda \otimes \mathscr{H}^*_\lambda)^{LH} = \mathrm{Hom}_{LH}(\mathscr{H}_\lambda, \mathscr{H}_\Lambda)\ .$$

These are precisely the states obtained from the more traditional description $\mathscr{H}_\Lambda = \oplus_\lambda \mathscr{H}_{\Lambda,\lambda} \otimes \mathscr{H}_\lambda$.

The observables of the coset CSGT are products of G and H Wilson operators subject to rules analogous to rules (a)–(c) of the previous section. First, the common center Z imposes the selection rule that λ and Λ transform the same way under Z. Second, the spectral flow associated to Z identifies pairs of representations: $\mathscr{H}_{\Lambda,\lambda} \cong \mathscr{H}_{\mu(\Lambda),\mu(\lambda)}$. Third, if the spectral flow fixes a pair of representations then $\mathscr{H}_{\Lambda,\lambda}$ is a reducible representation of the coset chiral algebra so there exist new operators. Generically rules (a)–(c) completely determine the representation content of coset theories, but there exists cases (e.g. conformal embeddings) with additional selection rules.

From the above description of Wilson lines it is clear that the braiding and modular transformation matrices of the coset theory must have the form $B_{G/H}=B_G\otimes B_H^{-1}$, $S_{G/H}=S_G\otimes S_H^{-1}$, and $T_{G/H}=T_G\otimes T_H^{-1}$ [21] subject to the above selection and identification rules.

The above ideas can be illustrated with many examples. For example the coset theory $U(1)_N\times U(1)_M/U(1)_{N+M}$ can be solved explicitly in both two and three dimensions and is equivalent to $U(1)_L$ where the level is $L=NM(N+M)/(N,M)^2$ (here (N, M) denotes the greatest common divisor) [#8]. Another famous example of a coset theory is that of parafermions of level N. These are equivalent to the CSGT based on $SU(2)_k\times U(1)_{-k}/\mathbb{Z}_2$. It is important to note that in the coset construction one mods out by the algebra of the rational torus, not just $\mathscr{A}(R)$. Rules (a), (b) reproduce the well-known description of the representations of parafermion algebras [22,23]. Similarly, the three famous $N=0$, 1, 2 discrete series can be studied from three dimensions. In the coset construction of the $N=0$ discrete series [24] one uses the gauge group $SU(2)_k\times SU(2)_1 \times SU(2)_{-(k+1)}/\mathbb{Z}_2$. The representations are labelled by spins (j_1, j_2, j_3) with $0\leqslant j_1\leqslant k/2$, $0\leqslant j_2\leqslant 1/2$ and $0\leqslant j_3\leqslant (k+1)/2$. The existence of a common center imposes the selection rule $j_1+j_2+j_3\in\mathbb{Z}$. Correspondingly there is an identification $(j_1, j_2, j_3)\sim(k/2-j_1, 1/2-j_2, (k+1)/2-j_3)$. We recover the traditional description [25] of fields $\phi_{p,q}$ defining $p=2j_1+1$, $q=2j_3+1$ and $m=(k+2)$. For the superdiscrete series we consider $SU(2)_k\times SU(2)_2 \times SU(2)_{-(k+2)}/\mathbb{Z}_2$, similar considerations apply except we must apply rule (c) above when k is even since the representation $(k/4, 1/2, (k+2)/4)$ is fixed by the spectral flow (in 2D this is the supersymmetric ground state.). Similarly, the $N=2$ discrete series corresponds to $SU(2)_k\times U(1)_2\times U(1)_{-(k+2)}/\mathbb{Z}_2$.

[#7] Jacobians are irrelevant here since they do not alter the canonical formalism we will derive.

[#8] When verifying our rules it is important to note that $\mathscr{A}(U(1))$ level N is in fact nonabelian and has a discrete center $\mathbb{Z}_{2N}$. Hence, the gauge group is $U(1)_N\times U(1)_M\times U(1)_{N+M}/\mathbb{Z}_{2(N,M)}$.

4. Orbifolds

We now turn to the CSGT for disconnected groups. For generic groups, this theory does not lead to any two-dimensional conformal field theory. However, a subset of the disconnected groups is interesting. Let G be a connected group with a discrete automorphism group P. Then one can constructed the semi-direct product group P⋉G. Quantizing the system on the disk and repeating the steps in the previous examples we find that the effective action is the WZW action for a field U on the boundary which takes values in G. The phase space is LG/G and leads to $\mathcal{A}(G)$, but because of P gauge invariance, the Hilbert space has to be truncated to the P invariant states (the states are in representations of P because P is an automorphism of G). This can be seen by considering the CSGT on $D\times S^1$. The functional integral in this case leads to the trace over the Hilbert space (since the hamiltonian of the 3D theory vanishes, this trace is infinite). In the functional integral we need to sum over P bundles. This sum projects out the states which are not P invariant. Therefore, $\mathcal{A}(P\ltimes G)=\mathcal{A}(G)/P$. This is the chiral algebra of the orbifold constructed as G/P.

By quantizing the system on the annulus N, we obtain all the representations of the orbifold model. The Hilbert space is a direct sum of products of two representations, one for each boundary of the annulus. Because of the sum over the P bundles we find the same representation of the orbifold algebra on the two boundaries [#9]. Indeed, the effective 2D theory on $\partial(N\times S^1)$ is the WZW theory summed over twisted boundary conditions in the "space" and "time" direction. This is known to lead to the orbifold.

Orbifolds and cosets are very similar in both two and three dimensions. In 3D we reduced the chiral algebra of the G theory by enlarging the gauge group. In 2D both theories are obtained by considering a G theory and gauging either a continuous subgroup, H/Z (to obtain G/H) or a discrete automorphism group, P (to obtain G/P). Finally note that the gauge group (G×H)/Z of the coset CSGT can also be written as (H/Z)⋉G which is the same as the prescription for orbifolds. In the classical limit of these theories the integral weight fields have a closed OPE. Therefore, there should be a one to one correspondence between these representations of the chiral algebra and representation spaces of some group [3]. This group is the gauge group of the 3D theory.

As examples consider the $c=1$ rational orbifolds [27,28]. From the above considerations [#10] it is clear that the appropriate gauge group is O(2). As a check let us count the number of representations on the torus. From conformal field theory one easily finds that for the rational orbifold of level N there are $N+7$ representations [28]. The moduli space of flat O(2) bundles splits into those which can and cannot be considered to be SO(2) bundles. The moduli space of flat SO(2) bundles is a torus, which must be divided by the $\mathbb{Z}_2$ action $z\to -z$. Of the $2N$ characters of the rational torus, $N+1$ are invariant [#11]. One then easily checks that in addition the phase space consists of six isolated points for twisted O(2) bundles. Each point contributes one quantum state for a total of $N+7$. As a second example consider the orbifold $SU(2)_k/\mathbb{Z}_2\times\mathbb{Z}_2$, where we take the quotient by 180° degree rotations around orthogonal axes. Here two interesting subtleties arise. First, the moduli space again decomposes into "twisted" and "untwisted" components (not to be confused with the 2D meaning of these words) but some of the twisted components are not points but manifolds. Second, some of the twisted components in fact contribute no quantum states, because of a global anomaly in the appropriate sector. One finds that the number of quantum states is $(11k+32)/2$ if k is even and $(11k+11)/2$ if k is odd, as can be derived by more conventional techniques [28].

5. Conjectured classification

We have explored a number of compact groups and have shown that all known chiral algebras can be ob-

[#9] This phenomenon is more general. The Hilbert space of the annulus for every gauge group is $\oplus_\lambda \mathcal{H}_\lambda \otimes \mathcal{H}_{\bar\lambda}$ where the sum is over all the irreps of $\mathcal{A}=\mathcal{H}_0$. The string is thus "thickened" to a ribbon. The left movers and the right movers "live" on different sides of this ribbon. This provides and explicit realization of a speculation in ref. [26].

[#10] E. Witten first suggested that O(2) would reproduce the rational orbifold. This suggestion motivated the above construction for orbifolds.

[#11] Or, dividing by $\mathbb{Z}_2$ produces a sphere, giving $N+1$ states by standard coadjoint quantization of SU(2).

tained from the quantization of a CSGT for a compact gauge group. Although CSGT's for noncompact groups exist and are interesting they should not lead to RCFT's. By definition, a conformal field theory is rational only if the number of conformal blocks on any compact surface Σ is finite. Therefore the 3D theory should assign a finite dimensional Hilbert space to Σ which is possible only if the classical phase space, the moduli space of flat G-connections on Σ, is compact. The moduli space is compact only when the underlying gauge symmetry is compact and hence RCFT can arise only from compact gauge groups.

Since the known RCFT's are so well organized by CSGT, we conjecture that *all chiral algebras of RCFT arise from the quantization of the 3D CSGT for some compact Lie group.* This conjecture is in accord with the philosophy of ref. [3] which emphasized that RCFT should be viewed as a generalization of group theory. If our conjecture is correct, the classification of chiral algebras is closely related to the classification of compact Lie groups. This correspondence is not one to one. Several chiral algebras can be obtained from more than one group. Also, as discussed earlier, some groups do not lead to a conformal field theory.

Although we do not have a proof of our conjecture, we will make it more plausible by suggesting an outline of a possible proof. In ref. [3] we proposed axioms for a RCFT. The classical version of these axioms arise in category theory and lead to group theory [29]. It is likely that a similar reconstruction theorem will show that the axioms in ref. [3] lead to quantum groups. In fact, using [4,5], one sees that every quantum group satisfies these axioms. Furthermore, the duality matrices of some quantum groups are the same as those of a corresponding RCFT.

If the reconstruction theorem is proven, the connection to the 3D CSGT theory can be established as follows. Consider the theory with gauge group G on $\mathbb{R}^2$ as space with sources at the point P_i. The sources can be represented as a copy of the coadjoint orbit G/T at each P_i [6]. The Hilbert space of the theory is $\otimes_i R_i$ where R_i is a representation space of G corresponding to the source at P_i. When the space is compactified to S^2, there are further restrictions on the Hilbert space – one more Gauss law has to be satisfied. In the semi-classical limit this Gauss law restricts the Hilbert space to the G-invariant tensors in $\otimes_i R_i$ [6]. It is possible (and the first non-trivial order in an expansion $1/k$ supports it) that at finite k the effect of this Gauss law is to restrict the Hilbert space to the invariant tensors of the quantum version of G. This will provide a conceptual explanation for the coincidence of the duality matrices of RCFT and $6j$ symbols for special quantum groups, and will complete the proof of our conjecture.

Acknowledgement

We would like to thank T. Banks, M. Bershadsky, R. Dijkgraaf, S. Elitzur, D. Friedan, P. Ginsparg, J. Harvey, H. Ooguri, A. Schwimmer, S. Shenker, E. and H. Verlinde, and especially E. Witten for discussions. This work was supported in part by NSF grant PHY-86-20266 and DOE contract DE-AC02-76ER02220.

References

[1] G. Moore and N. Seiberg, Nucl. Phys. B 313 (1989) 16.
[2] R. Dijkgraaf and E. Verlinde, presented Annecy Conf. on Conformal field theory.
[3] G. Moore and N. Seiberg, IAS preprint IASSNS-HEP-88/39, Commun. Math. Phys., in press.
[4] N. Reshetikhin, LOMI preprints E-4-87, E-17-87; A.N. Kirillov and N. Reshetikhin, LOMI preprint E-9-88.
[5] L. Alvarez-Gaumé, C. Gomez and G. Sierra, CERN preprint CERN-TH-5267/88.
[6] E. Witten, IAS preprint IASSNS-HEP-88/33.
[7] D. Friedan and S. Shenker, Nucl. Phys. B 281 (1987) 509; D. Friedan, Phys. Scr. T 15 (1987) 72.
[8] E. Witten, Commun. Math. Phys. 92 (1984) 455.
[9] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, to appear.
[10] N. Pressley and G. Segal, Loop groups (Oxford U.P., Oxford, 1986).
[11] G. 't Hooft, Nucl. Phys. B 138 (1978) 1.
[12] A. Alekseev, L. Fadeev and S. Shatashvili, J. Geom. Phys., in press.
[13] P. Goddard and D. Olive, Intern. J. Mod. Phys. A 1 (1986) 303.
[14] D. Bernard, Nucl. Phys. B 288 (1987) 628.
[15] G. Felder, K. Gawedzki and A. Kupiainen, Commun. Math. Phys. 117 (1988) 127.
[16] E. Verlinde, Nucl. Phys. B 300 (1988) 360.
[17] D. Gepner and E. Witten, Nucl. Phys. B 278 (1986) 493.
[18] M.F. Atiyah and R. Bott, Phil. Trans. R. Soc. Lond. A 308 (1982) 523.
[19] R. Brustein, S. Yankielowicz and J.-B. Zuber, preprint TAUP-1647-88.

[20] W. Nahm, Duke Math. J. 54 (1987) 579;
K. Bardakci, E. Rabinovici and B. Saering, Nucl. Phys. B 299 (1988) 151;
K. Gawedzki and A. Kupiainen, preprints HU-TFT-88-29;-34; IHES/P/88/45;
A.M. Polyakov, Les Houches Lectures;
D. Karabali, Q.-H. Park, H.J. Schnitzer and Z. Yang, preprint BRX-TH-247.

[21] M.R. Douglas, PhD thesis;
B. Blok and S. Yankielowicz, CERN preprint CERN-TH-5250/88.

[22] V.A. Fateev and A.B. Zamolodchikov, Sov. Phys. JETP 62 (1985) 215.

[23] D. Gepner and Z. Qiu, Nucl. Phys. B 285 (1987) 423.

[24] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103 (1986) 105.

[25] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333;
D. Friedan, Z. Qiu and S. Shenker, in: Vertex operators in mathematics and physics, eds. J. Lepowsky et al. (Springer, Berlin, 1984); Phys. Rev. Lett. 52 (1984) 1575.

[26] E. Witten, Phys. Rev. Lett. 61 (1988) 670.

[27] P. Ginsparg, Nucl. Phys. B 295 (1988) 153;
R. Dijkgraaf, E. Verlinde and H. Verlinde, Commun. Math. Phys. 115 (1988) 649.

[28] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, IAS preprint IASSNS-HEP-88-47.

[29] P. Deligne and J.S. Milne, Lecture Notes Mathematics, No. 900 (Springer, Berlin, 1982);
P. Deligne, Catégories tannakiennes, IAS preprint.