Three-dimensional quantum geometry and black holes

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Abstract

We review some aspects of three-dimensional quantum gravity with emphasis in the 'CFT $\to$ Geometry' map that follows from the Brown-Henneaux conformal algebra. The general solution to the classical equations of motion with anti-de Sitter boundary conditions is displayed. This solution is parametrized by two functions which become Virasoro operators after quantisation. A map from the space of states to the space of classical solutions is exhibited. Some recent proposals to understand the Bekenstein-Hawking entropy are reviewed in this context. The origin of the boundary degrees of freedom arising in 2+1 gravity is analysed in detail using a Hamiltonian Chern-Simons formalism.

1 Introduction

General relativity is a highly complicated non-linear field theory both classically and quantum mechanically. Even though a large number of classical solutions exists, a general classification of the space of solutions has never been achieved. The non-renormabizibility of quantum gravity is not related to this issue, but if the general structure of the space of solutions of the Einstein equations was known, then the quantum version of phase space perhaps would be more manageable.

It is this aspect of three-dimensional gravity that makes it attractive because the general solution to the equations of motion can be written down.

In this paper, we shall exploit this fact trying to formulate a quantum theory of black holes by quantizing the space of solutions directly.

Another important aspect of three-dimensional gravity is its formulation as a Chern-Simons theory[1]. Quantum Chern-Simons theory is well understood for compact groups[2, 3, 4]. However, we shall be interested in Euclidean gravity with a negative cosmological constant whose associated group is $SL(2, C)$, which is not compact. The quantization is then not straightforward. We shall follow an alternative route by first solving the equations of motion with prescribed boundary conditions and then quantise. We shall see that the boundary conditions will play an important role in making the quantum theory well-defined.

1.1 Brief description of the results contained in this article

Let us start by briefly mentioning, without proofs, the main results which will be of interest for us here. The relevant proofs will be given below. We should remark at this point that most of the results presented here are known in the literature in various contexts (the relevant quotations will be given in the main text). The aim of this article is to put things together in a self-contained framework, and to explore some aspects of quantum black holes in three dimensions.

Let M be a three dimensional manifold with a boundary denoted by ∂M. We assume that ∂M has the topology of a 2-torus. Let $\{w, \bar{w}, \rho\}$ coordinates on M such that the boundary is located at $e^\rho =: lr \to \infty$, and $w = \varphi + it$, $\bar{w} = \varphi - it$ are complex coordinates on the torus. The three-dimensional metric [1],

$$ds^2 = 4Gl(Ldw^2 + \bar{L}d\bar{w}^2) + \left(l^2 e^{2\rho} + 16G^2 L\bar{L}e^{-2\rho}\right) dw d\bar{w} + l^2 d\rho^2, \qquad (1)$$

where $L = L(w)$ and $\bar{L} = \bar{L}(\bar{w})$ are arbitrary functions of their arguments satisfies the following properties:

(i) Exact solution
The metric (1) is an exact solution to the three-dimensional vacuum Einstein equations with a negative cosmological constant $\Lambda = -1/l^2$. The leading

[1] G represents Newton's constant and l is the anti-de Sitter radius related to a negative cosmological constant by $\Lambda = -1/l^2$. We set $\hbar = 1$ throughout the paper.

and first subleading terms of (1) (in powers of $r = e^\rho$) are, of course, the ones dictated by the general analysis of [5] for asymptotically anti-de Sitter spacetimes. What is perhaps not so well-known is that adding the term $e^{-2\rho}L\bar{L}$, the metric becomes an *exact* solution. Most importantly, (1) is the most general solution, up to trivial diffeomorphisms, which is asymptotically anti-de Sitter. Note that since (1) contains two arbitrary functions, it gives rise to an infinite number of solutions.

We shall work in the Euclidean sector of the theory. This means that w is a complex coordinate related to the spacetime coordinates as $w = \varphi + it$. The metric (1) is then complex. As we shall see, this will not bring in any problems in the quantization. For real values of w, the metric (1) is a solution to the Minkowskian equations of motion.

(ii) Physical degrees of freedom

Two solutions of the form (1) with different values for L and $\bar{L}$ represent physically different configurations which cannot be connected via a gauge transformation. In the quantum theory, where L and $\bar{L}$ will become operators, different expectations values for them will be associated to different solutions.

This is a non-trivial statement. Since (1) is a solution to the three-dimensional Einstein equations it has constant curvature and then, locally, is isometric to anti-de Sitter space (see Eq. (11) below). The point here is that the coordinate transformations which change the values of L and $\bar{L}$ are not generated by constraints and therefore they are not gauge symmetries. This point will be analysed in detail in the Chern-Simons formulation in section 2.3, and in the metric formulation in Sec. 4.2.

(iii) Residual conformal symmetry

The metric (1) has a residual conformal symmetry. There exists a change of coordinates $\{w, \bar{w}, \rho\} \to \{w', \bar{w}', \rho'\}$ such that the new metric looks exactly like (1) with new functions L' and $\bar{L}'$. See Sec. 4.2 for the proof of this statement. This change of coordinates is parametrized by two functions $\varepsilon(w)$ and $\bar{\varepsilon}(\bar{w})$. The new function L' is related to the old one via $L' = L + \delta L$ with,

$$\delta L = i(\varepsilon \partial L + 2\partial\varepsilon L - \frac{c}{12}\partial^3\varepsilon) \tag{2}$$

where c is given by

$$c = \frac{3l}{2G}. \tag{3}$$

The same transformation holds for $\bar{L}$. Thus, under this symmetry, L and $\bar{L}$ are quasi-primary fields of conformal dimension 2. This symmetry, properly defined acting on the gravitational variables, can be shown to be also a global symmetry of the action[5]. The canonical generators are the functions L and $\bar{L}$ themselves and the associated algebra is the Virasoro algebra[5],

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n^3\delta_{n+m}, \tag{4}$$

where the central charge is defined in (3) and,

$$L(w) = \sum_{n\in Z} L_n e^{inw}. \tag{5}$$

We are using here a non-standard form of the central term. The usual $n(n^2-1)$ form can be obtained simply by shifting the L_0 mode as $L_0 \to L_0 - c/24$. This convention is appropriated to the black hole background which has an exact $SO(2) \times SO(2)$ invariance. See [6] for a discussion on this point in the supergravity context.

(iv) Asymptotic conformal symmetry

The residual symmetry (2) is not an exact symmetry of any background metric. Rather it is a symmetry of the space of solutions described by (1); it maps one solution into another one. However, this symmetry can be regarded as an *asymptotic* symmetry of anti-de Sitter space because the $r \to \infty$ form of (1) is asymptotic Euclidean adS_3 space (note the redefinition of coordinates: $w = \varphi + it$, $r = le^\rho$),

$$ds^2_{(r\to\infty)} \to r^2(dt^2 + d\varphi^2). \tag{6}$$

(We have kept here only the leading terms in powers of r, but note that there is also a term $2i(L-\bar{L})dtd\varphi$ of order one which is allowed by the boundary conditions[5].) Since the asymptotic behaviour (6) does not see L and $\bar{L}$ it is invariant under the transformation (2). This symmetry was discovered in [5]. See [7, 8] for recent discussions.

(v) Basic dynamical variables and induced Poisson brackets

Up to some global issues, the Virasoro algebra (4) can be regarded as the basic

Poisson bracket algebra of the gauge-fixed residual variables. In other words, the functions $L(w)$ and $\bar{L}(\bar{w})$ appearing in (1) are the part of the metric field $g_{\mu\nu}(x^\mu)$ which survives after the gauge is fixed (i.e., after gauge conditions are imposed and the constraints solved), with anti-de Sitter boundary conditions. The equal-time Poisson bracket of general relativity, $\{\pi^{ij}, g_{kl}\} = \delta^{ij}_{kl}$, induces the Virasoro algebra (4) on the residual dynamical functions L and $\bar{L}$. A technical note is convenient here. From the dynamical point of view, L and $\bar{L}$ both depend on w and $\bar{w}$. However, the gauge-fixed equations of motion read $\partial_{\bar{w}} L = 0$ and $\partial_w \bar{L} = 0$ leading to $L = L(w)$ and $\bar{L} = \bar{L}(\bar{w})$.

The identification of the Virasoro operators as basic variables is not natural from the point of view of conformal field theory. In the standard situation, the Virasoro algebra is associated to a symmetry rather than to the basic commutator. (A good analogy is to consider the angular momentum components L_i as basic variables, satisfying $[L_i, L_j] = i\epsilon_{ijk} L_k$, without knowing the existence of q^i, p_j.) One of the main problems of three-dimensional quantum gravity is to identify what is the conformal field theory behind the Brown-Henneaux conformal symmetry. Classically, Liouville theory [9] seems to be a good candidate, however, its quantisation does not give the right counting for the black hole entropy degeneracy[10]. Treating the Virasoro algebra as basic Poisson algebra is also well-motivated classically but it does not give the right counting either (see Sec. 4.4). We shall discuss this issues in Sec. 4.4, as well as two possible modifications of the boundary dynamics which do provide the right counting of states.

(vi) Black holes and adS space

The space of solutions described by (1) contains black holes. If L and $\bar{L}$ are constants (no $w, \bar{w}$ dependence) with only L_0, $\bar{L}_0$ different from zero and parametrized as

$$Ml = L_0 + \bar{L}_0, \qquad J = L_0 - \bar{L}_0, \tag{7}$$

then the metric (1) is globally isometric to the Euclidean three-dimensional black hole [11, 12] of mass M and angular momentum J. Eq. (7) means that the Virasoro operators vanish on the vacuum black hole. The corresponding algebra is (4). The Euclidean black hole metric in Schwarzschild coordinates reads

$$ds^2 = l^2 N^2 dt^2 + N^{-2} dr^2 + r^2 (d\varphi + i N^\varphi dt)^2, \tag{8}$$

with

$$N^2(r) = -8MG + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}, \tag{9}$$

$$N^{\varphi}(r) = \frac{4GJ}{r^2}. \tag{10}$$

See Sec. 4.3 for the explicit transition from (1) to (8).

For $M > |J|$, this metric has two horizons which are the solutions to the equation $N^2(r_\pm) = 0$. It is often convenient to define the Euclidean angular momentum J_E as $J_E = iJ$ and then the i in (8) does not occur. Note also that in the Euclidean sector, the black hole manifold does not see the interior $r < r_+$. See Sec. 4.3 for more details on the relation between (8) and (1). Note that since the coordinates w and $\bar{w}$ are defined on a torus, the only globally well-defined solutions are the ones with constant L and $\bar{L}$.

For $J = 0$ and $8MG = -1$ the metric (8) reduces to Euclidean anti-de Sitter space in three dimensions,

$$ds^2_{adS} = l^2\left(1 + \frac{r^2}{l^2}\right)dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\varphi^2. \tag{11}$$

(vii) A quantum metric, the 'CFT→ Geometry' map and black hole entropy

Once the functions L and $\bar{L}$ are promoted to be operators acting on Fock space, the metric (1) becomes a well-defined operator, denoted as $d\hat{s}^2$, on that space. We then find a map from Fock's space (representations of the Virasoro algebra) into the independent classical solutions of Einstein's equations. Let $|\Psi>$ a state in Fock's space, we have

$$|\Psi> \;\rightarrow\; ds^2_\Psi =<\Psi|d\hat{s}^2|\Psi> \tag{12}$$

with $L_\Psi =<\Psi|L|\Psi>$ and $\bar{L}_\psi =<\Psi|\bar{L}|\Psi>$. By construction ds^2_Ψ is a solution to the classical Einstein equations because (1) is a solution for arbitrary functions L and $\bar{L}$. It also follows that the full set of states $|\Psi>$ generates the full space of classical solutions. Note that this map is valid independently of the structure of the conformal field theory generating the Virasoro algebra. We can then ask the question of how many states are there in Fock space such that they induce through (12) a black hole of a mass M

and angular momentum J. The answer to this question of course depends on the structure of the Hilbert space. We shall study this point in detail in Sec. 4.4.

(viii) Relation to 2d induced gravity

Finally, note that for $\bar{L} = 0$ and fixed r, the metric (1) is equal to Polyakov's [13] 2d lightlike metric which yields an $SL(2, \Re)$ algebra. Since 3d gravity is known to induce 2d gravity at the boundary (r fixed), the understanding of the quantum properties of (1) may yield new information about 2d gravity.

1.2 Organisation of the article

The goal of this article is to discuss and provide the proofs for the above properties of the metric (1). We have written (i)-(viii) in a metric formulation of gravity because our final target is quantum gravity. However, the explicit proofs will be given in terms of the Chern-Simons formulation [1] of three-dimensional gravity because they are simpler and provide a rich mathematical structure.

In Sec. 2 we give a short introduction to Chern-Simons gravity and its phase space. A detailed discussion about boundary degrees of freedom is included in that section. In Sec. 3 the explicit solution to the equations of motion, with two different classes of boundary conditions, is written down (in terms of the Chern-Simons fields) and their induced Poisson brackets are displayed. Finally, in Sec. 4, we go back to the metric formulation and apply the results to quantum three-dimensional gravity.

2 Chern-Simons gravity and global degrees of freedom

In this section we shall first briefly describe the Chern-Simons formulation of three-dimensional gravity. Then we analyse the issue of global degrees of freedom associated to the presence of boundaries. We shall also show in this section (see Sec. 2.6) how the boundary conditions solve part of the unitary problems of three-dimensional gravity.

2.1 Chern-Simons gravity, its equations of motion and their solutions

In our approach to the quantum black hole problem, the Chern-Simons formulation of 2+1 gravity will be of great help. This formulation was discovered in [1] and its quantum properties (for closed manifolds) were explored in [14]. An extensive treatment can be found in [15]. In a few words, the Chern-Simons formulation is a field redefinition that simplifies the equations and introduces a rich mathematical structure.

The basic variables of general relativity in the tetrad formalism are the triad e^a and the spin connection[2] ω^a. The equations of motion of three dimensional gravity with a negative cosmological constant in these variables are simply

$$R^a = \frac{1}{2l^2}\epsilon^a{}_{bc}\, e^b{\scriptstyle\wedge} e^c, \qquad T^a = 0. \tag{13}$$

We define now two new fields according to,

$$A^a = \omega^a + \frac{i}{l}e^a, \qquad \bar{A}^a = \omega^a - \frac{i}{l}e^a. \tag{14}$$

The 1-form A^a is an $SL(2,C)$ Yang-Mills gauge field. Let F^a and $\bar{F}^a$ the curvatures associated to A^a and $\bar{A}^a$. The discovery of Achúcarro and Townsend [1] is that the equations,

$$F^a = 0, \qquad \bar{F}^a = 0, \tag{15}$$

are exactly equivalent to the three-dimensional Einstein equations (13). Furthermore, the Einstein-Hilbert action is equal to the combination,

$$I[A,\bar{A}] = I[A] - I[\bar{A}], \tag{16}$$

where $I[A]$ is the Chern-Simons action,

$$I[A] = \frac{k}{4\pi}\int \mathrm{Tr}(AdA + \frac{2}{3}A^3). \tag{17}$$

[2]In three dimensions one defines $\omega^a = (-1/2)\epsilon^a{}_{bc}\omega^{bc}$. It follows that the 2-form curvature $R^{ab} = d\omega^{ab} + \omega^a{}_c{\scriptstyle\wedge}\omega^{cb}$ can be written in the form $R^{ab} = -\epsilon^{ab}{}_c R^c$ with $R^a = d\omega^a + (1/2)\epsilon^a{}_{bc}\omega^b{\scriptstyle\wedge}\omega^c$. In the same way, the torsion $T^a = de^a + \omega^a{}_b{\scriptstyle\wedge} e^b$ reads $T^a = de^a + \epsilon^a{}_{bc}\omega^b{\scriptstyle\wedge} e^c$. These definitions depend on the signature. The formulae displayed here are appropriated to Euclidean signature.

To determine the Chern-Simons coupling constant, or level, k as a function of the gravitational constants G and l we need to fix the representation of A and $\bar{A}$. We use the anti-Hermitian $SU(2)$ generators,

$$J_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{18}$$

which satisfy $[J_a, J_b] = \epsilon_{ab}{}^c J_c$ and $\mathrm{Tr}(J_a J_b) = -(1/2)\delta_{ab}$ and define

$$A = A^a J_a, \quad \bar{A} = \bar{A}^a J_a. \tag{19}$$

Note that we use the same J's in both cases. This means that $\bar{A}$ is not the complex conjugate of A. With these conventions, comparing the Chern-Simons action with the Einstein-Hilbert action one finds

$$k = -\frac{l}{4G}. \tag{20}$$

The sign of k depends on the identity $\sqrt{g} = \pm e$ where e is the determinant of the triad. This sign determines the relative orientation of the coordinate and orthonormal basis. We have chosen here the plus sign which means that we work with $e > 0$.

Note that the gauge field A is complex and thus the relevant group is $SL(2, C)$ which is non-compact. This means that we cannot apply in a straightforward way the quantization of Chern-Simons theory described in [2]. Our prescription to define the quantum theory will be to first find the general solution the classical equations of motion, under prescribed boundary conditions, and then quantize that space. As we shall see, the boundary conditions will play a key role in making the quantum theory unitary (see Sec. 2.6).

The convenience of the Chern-Simons formulation is evident. Instead of working with a second order action in terms of the metric, we work with two flat Yang-Mills fields. The equations (15) show clearly that 2+1 gravity does not have any local degrees of freedom. This means that all dynamics is contained in the holonomies [14] and boundary degrees of freedom [16, 17].

The general solution to the equations (15) can be written in the form,

$$A = g^{-1}dg + g^{-1}Hg, \tag{21}$$

where H is also flat ($dH + H{\wedge}H = 0$) but cannot be written as $u^{-1}du$ with u single valued. Similar arguments hold for $\bar{A}$. The group element $g(x)$ is a single valued map from the manifold to the group. The space of solutions (21) is invariant under

$$A \to A' = U^{-1}AU + U^{-1}dU \tag{22}$$

where U is another map from the manifold to the group. In principle (see below for a detailed discussion), we can use this symmetry to set $g = 1$ in (21) and thus all solutions are classified only by the independent values of holonomy H. The quantization of this sector of phase space was first discussed in [14]. Its dimensionality is finite and cannot account for the large black hole degeneracy. For this reason we shall not consider them here anymore. However, it is important to stress that the black hole gauge field does have non-trivial holonomies. Indeed, it can be shown that the gauge field corresponding to a black hole satisfies [18],

$$\mathrm{P}\exp\oint A =: \exp(w), \qquad \mathrm{P}\exp\oint \bar{A} =: \exp(\bar{w}) \tag{23}$$

where

$$\mathrm{Tr}(w^2 + \bar{w}^2) = 32\pi^2 MG, \qquad \mathrm{Tr}(w^2 - \bar{w}^2) = \frac{32\pi^2 JG}{l}, \tag{24}$$

and M and J are the black hole mass and angular momentum, respectively. Only for $8GM = -1$ and $J = 0$ these holonomies are trivial. The corresponding solution is anti-de Sitter space (11).

2.2 The strategy

Since Chern-Simons theory does not have any local excitations, all relevant degrees of freedom are global. We shall consider here the situation on which the topology is fixed and thus all relevant states come from the presence of boundaries. The existence of boundary degrees of freedom has been analysed in great detail by Carlip [19] using a covariant formalism and path integrals (see also [17] for an approach similar to ours). Here, we shall describe an equivalent procedure based on the Hamiltonian formalism in the form discussed by Regge and Teitelboim [20]. This approach can be summarised in the following steps. Given an action $I[\phi]$ with a gauge symmetry $\delta_G\phi$ we need to:

- Impose boundary conditions on the fields such that $\delta I[\phi]/\delta\phi$ exists. These boundary conditions are not unique and their election represent an important physical input into the theory. [In practice, one first decides the boundary conditions and then add to the action the necessary boundary terms to make it differentiable.]

- Find the sub-group of gauge transformations that leave the boundary conditions and the action invariant.
- Find the canonical generators which generate the symmetries of the action. If a generator is a constraint, we shall call the associated symmetry a *gauge* symmetry. Configurations which differ by gauge symmetries are identified and represent the same physical state. Conversely, if a generator is different from zero (even on-shell), we call the associated symmetry a *global* symmetry. Note that according to this definition, global symmetries do not need to be rigid. Global symmetries map the space of physical states into itself.

We shall see that a proper distinction between global and gauge symmetries is crucial to understand the boundary degrees of freedom in Chern-Simons theories.

2.3 Global symmetries and boundary degrees of freedom

The appearance of boundary degrees of freedom can be summarized as follows. Chern-Simons theory has $3N$ fields A^a_μ ($\mu = 0, 1, 2; \quad a = 1, ..., N$). However, the gauge symmetries of the action tells us that they do not represent independent physical degrees of freedom. In fact, locally, using the symmetry (22) one can kill all of them (the temporal component A^a_0 is a Lagrange multiplier, while the spatial components A^a_i are $2N$ fields subject to N constraints $F^a_{ij} = 0$ plus N gauge conditions).

The question we want to address here is whether the symmetry (22) is really a gauge symmetry, in the sense that two fields related by it are to be considered the same, or not. After properly defining what a gauge transformation is, we shall see that at the boundary the transformation (22) is not a gauge symmetry, although it is still a symmetry of the action. Then, two solutions of the form (21) with g and g' such that, at the boundary, $g \neq g'$ represent two different physical configurations. This boundary effect can give rise to an infinite number of degrees of freedom (independent solutions to the equations of motion). At this point we can make contact with Carlip's would-be-gauge degrees of freedom approach: the field g, at the boundary, is dynamical and its dynamics is governed by a WZW action[19].

In the presence of boundaries the definition of a gauge symmetry becomes delicate because not all the transformations encoded in (22) are generated by constraints. Indeed, if U does not approach the identity map at the boundary, then the associated canonical generator is a non-zero quantity and hence that transformation is not a gauge symmetry. See Sec. 2.5 for a proof of this statement in Chern-Simons theory.

Following Dirac's quantization procedure (see [21] for an extensive treatment), we define a *gauge* transformation as a symmetry generated by a (first class) constraint. On the contrary, a symmetry of the action generated by a non-zero quantity is called *global*, even if it is not rigid. By definition, the space of physical states, or phase space, is the set of fields which satisfy the equations of motion, modulo gauge transformations. Let us ignore the holonomies for a moment. The general solution (21) then reduces to $A = g^{-1}dg$. If no boundaries are present, this space of solutions is trivial containing only one element $A = 0$ because the transformations (22) are generated by constraints (hence, they represent gauge symmetries) and one can use (22) to set $g = 1$ and thus $A = 0$.

On the contrary, if there is a boundary, part of the symmetry (22) is not generated by a constraint (to be proved below). Therefore, while it is still true that we can transform any flat A to 0 using (22), it is not true that the state A and the state 0 represent the same physical configuration. Both states, $A \neq 0$ and $A = 0$ (at the boundary), are solutions to the equations of motion and they are related by a symmetry of the action. However, they are physically distinguishable. Indeed, there is a gauge invariant conserved charge which takes different values in each state. Our main problem is then to determine the set of fields $\hat{A}$ which solve the equations of motion and cannot be set to zero by the action of a constraint. As we shall see, in Chern-Simons theory there is an infinite number of them.

In a quantum mechanical notation, the above discussion can be summarized as follows. Denote by G_0 the set of transformations which are true gauge symmetries generated by constraints, and by Q those which are not. Physical states satisfy $G_0|\Psi >= 0$. On the other hand, Q generates a symmetry of the space of physical states, that is $Q|\Psi >= |\Psi' >$. We shall prove explicitly (at least in Chern-Simons theory; for a general discussion see [22]) that G_0 and Q satisfy an algebra of the form,

$$[G_0, G_0] = G_0, \tag{25}$$

$$[G_0, Q] = G_0, \tag{26}$$

$$[Q, Q] = Q + c \tag{27}$$

where c represents (schematically) a possible central term. Eq. (25) is the definition of first class constraints. Eq. (26) means that if $|\Psi>$ is physical ($G_0|\Psi>=0$) then $Q|\Psi>$ is also physical (Q generates a global symmetry of the Hilbert space). Finally, Eq. (27) is the algebra of the globally symmetry. The appearance of central terms in (27) cannot be discarded by a general principle [22]. Note however that since Q does not generate a gauge symmetry and it is different from zero, the central term does not represent any trouble after quantization. An interesting and important example on which the central term is present was discovered in [5].

The above discussion is a quick summary of the results presented in [20, 23, 22, 5], and many other papers that have followed this work. The nice property of Chern-Simons theory is that these ideas can be tested with minimum calculations. Another system which is simple to analyse is Yang-Mills theory on which the above analysis leads to the definition of global colour charges[24]. However, in that case, the resulting global algebra is finite dimensional and does not have any central terms.

2.4 Boundary conditions in Chern-Simons gravity

2.4.1 Making the action differentiable

The black hole manifold is asymptotically anti-de Sitter and then it has a boundary. In the Euclidean sector, the boundary has the topology of a torus with compact coordinates φ and t. It is convenient to define the complex coordinates on the torus

$$w = \varphi + it, \qquad \bar{w} = \varphi - it, \tag{28}$$

and then $A_\varphi d\varphi + A_t dt = A_w dw + A_{\bar{w}} d\bar{w}$.

Boundary conditions are necessary in order to ensure that the action principle has well defined variations. As discussed above, all the dynamics of 2+1 gravity is contained in the boundary conditions. For this reason, it is a key problem to choose them judiciously. In particular, if they are too strong there will be no dynamics left in the theory. For the black hole problem (which is asymptotically anti-de Sitter) there is a natural choice of boundary conditions first discussed in [9] in the Minkowskian signature and extended to Euclidean signature in [25]. In the coordinates (28) they read simply

$$A^a_{\bar{w}} = 0, \qquad \bar{A}^a_w = 0 \qquad \text{(at the boundary).} \tag{29}$$

A quick way to convince ourselves that the black hole satisfies this condition is to consider the constant curvature metric

$$ds^2 = e^{2\rho}(dx^2 + dy^2) + l^2 d\rho^2. \tag{30}$$

A natural election for the triads is $\{e^1 = e^\rho dx, e^2 = e^\rho dy, e^3 = l d\rho\}$. The torsion equation $de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c = 0$ yields for the components of ω^a $\{\omega^1 = -(1/l)e^\rho dy,\ \omega^2 = (1/l)e^\rho dx, \omega^3 = 0\}$. Defining $w = x + iy$ it is clear that $A^a = \omega^a + (i/l)e^a$ and $\bar{A}^a = \omega^a - (i/l)e^a$ satisfy (29). It can be shown that the black hole metric (8) which is also of constant curvature satisfies (29) as well [25]. See [26] for the explicit transition from (30) to (8).

This example also illustrates the choice of orientation. Suppose we choose a new set of triads given by $\tilde{e}^1 = -e^\rho dx$, $\tilde{e}^2 = -e^\rho dy$ and $\tilde{e}^3 = -l d\rho$. These new fields satisfy the torsion equation because is homogeneos in e, and Einstein equations because they are quadratic in e. However, the determinant of $\tilde{e}^a_\mu$ is negative. As we remarked before, the value of k given in (20) depends on the orientation of the orthonormal basis, and the identity $\sqrt{g} = \pm e$. We have chosen $e > 0$ and then the election $\tilde{e}^a$ is not allowed.

Let us check that (29) are enough to make the action differentiable. The variation of the Chern-Simons action gives a term proportional to the equations of motion plus a boundary term,

$$\begin{aligned} \delta I_{CS} &= \int_M (\text{eom})_a \delta A^a + \frac{k}{4\pi} \int_{\partial M} g_{ab} A^a \wedge \delta A^b \\ &= \int_M (\text{eom})_a \delta A^a + \frac{k}{4\pi} \int_{\partial M} g_{ab} (A^a_w \delta A^b_{\bar{w}} - A^a_{\bar{w}} \delta A^b_w) \\ &= \int_M (\text{eom})_a \delta A^a + 0. \end{aligned} \tag{31}$$

The boundary term vanishes due to (29). Thus, the variation of the action under the boundary condition (29) is well defined. Later we will restrict further the values of the gauge field at then boundary, but for the purposes of this discussion the above boundary conditions are very useful.

In summary, we work with the Chern-Simons action (16) supplemented with the boundary conditions (29) and no added boundary terms. [Note that when passing to the Hamiltonian formalism there will be a boundary term[25].] As a further check that this action is appropriated to the black hole problem, one can prove [27] that its value on the Euclidean black hole solution is finite and gives the right canonical free energy (Gibbons-Hawking approximation).

2.4.2 The chiral boundary group

The second step in the Regge-Teitelboim procedure is to determined how the gauge symmetries are affected by the boundary conditions, i.e., to determine the residual group of transformations that preserves (29). This is actually very simple. We look for the set of parameters λ^a satisfying

$$\delta A^a_{\bar{w}} = D_{\bar{w}} \lambda^a = 0 \qquad \text{(at the boundary)}. \tag{32}$$

Since by (29) $A_{\bar{w}} = 0$ this condition simply imply that $\partial_{\bar{w}} \lambda^a = 0$. The subset of gauge transformations leaving (29) invariant are then those whose parameters at the boundary are chiral, only depend on w.

Let us now check that this group leaves the action invariant. The variation of the Chern-Simons action under $\delta A^a = D\lambda^a$ gives a boundary term,

$$\begin{aligned} \delta I_{CS} &= \frac{k}{4\pi} \int_{\partial M} A^a \wedge D\lambda_a \\ &= \frac{k}{4\pi} \int_{\partial M} (A^a_w D_{\bar{w}} \lambda_a - A^a_{\bar{w}} D_w \lambda_a) \\ &= 0 \end{aligned} \tag{33}$$

which vanishes thanks to (29) and (32). There is an important point to be stressed here. It is often said in the literature that the Chern-Simons action is invariant under $\delta A^a = D\lambda^a$ only if $\lambda = 0$ at the boundary. This is, as we have just shown, not true. The right statement is that λ cannot be completely arbitrary at the boundary but it can be different from zero. Under the boundary condition (29), the action is invariant under transformations with non-zero values of λ^a at the boundary provided that parameter is chiral ($\lambda^a = \lambda^a(w)$). This gives rise to an infinite dimensional symmetry.

2.5 Affine (Kac-Moody) algebras

Let us briefly describe the main steps leading to (25-27) in Chern-Simons theory. For more details, the reader is referred to [28, 29].

In the 2+1 decomposition of the gauge field $A^a = A^a_0 dt + A^a_i dx^i$, the Chern-Simons action reads,

$$I[A_i, A_0] = \frac{k}{8\pi} \int dt \int_\Sigma \epsilon^{ij} \delta_{ab} (A^a_i \dot{A}^b_i - A^a_0 F^b_{ij}) + B, \tag{34}$$

where B is a boundary term. Here we have used that $\mathrm{Tr}(J_a J_b) = -(1/2)\delta_{ab}$. The coordinates x^i are local coordinates on the spatial surface denoted by Σ. This action has $2N$ dynamical fields A^a_i ($a = 1, ..., N$; $i = 1, 2$) and N Lagrange multipliers A^a_0. The dynamical fields satisfy the basic equal-time Poisson bracket algebra,

$$\{A^a_i(x), A^b_j(y)\} = \frac{4\pi}{k} \epsilon_{ij} \delta^{ab} \delta^2(x, y). \tag{35}$$

The Poisson bracket of two functions $F(A_i)$ and $H(A_i)$ is computed as

$$\{F, H\} = \frac{4\pi}{k} \int_\Sigma d^2 z \frac{\delta F}{\delta A^a_i(z)} \epsilon_{ij} \delta^{ab} \frac{\delta H}{\delta A^b_j(z)}. \tag{36}$$

The functionals F and H need to be differentiable with respect to A_i.

The equation of motion with respect to A_0 leads to the constraint equation,

$$G^a_0 = \frac{k}{8\pi} \epsilon^{ij} F^a_{ij} \approx 0, \tag{37}$$

which, we expect, will be the canonical generator of the gauge transformations $\delta A^a_i = D_i \lambda^a$. This is indeed true but only for those transformation whose parameters vanish at the boundary. Indeed, define $G_0(\lambda) = \int_\Sigma \lambda_a G^a_0$ and compute $\delta_\lambda A_i(x) = [A^a_i(x), G_0(\lambda)]$. It is direct to see that the functional derivative of $G_0(\lambda)$ with respect to A_i is well-defined only if λ^a vanishes at the boundary. In that case, one does find $[A^a_i(x), G_0(\lambda)] = D_i \lambda^a$ and thus $G_0(\lambda)$ generates the correct gauge transformation. (We stress here that $G_0(\lambda)$ should not be identified with the constraint: $G_0(\lambda)$ is the constraint smeared with a parameter that vanishes at the boundary.)

However, as we discussed in Sec. 2.4.2, the Chern-Simons action with the boundary condition (29) is also invariant under transformations whose parameters at the boundary are chiral $\lambda^a = \lambda^a(w)$ but different from zero. What is then the generator of those transformations? Consider

$$Q(\lambda) = \int_\Sigma \lambda_a G_0^a - \frac{k}{4\pi} \int_{\partial\Sigma} \lambda_a A^a. \tag{38}$$

It is easy to check that the boundary term arising when varying the bulk part of (38) is cancelled by the boundary term, without imposing any conditions over λ. The combination (38) then has well defined variations even if λ does not vanish at the boundary. Furthermore, one can check that $[A_i^a(x), Q(\lambda)] = D_i\lambda^a$ and therefore Q indeed generates those transformations whose parameters do not vanish at the boundary.

The key point here is that Q is no longer a combination of the constraints (for $\lambda^a|_{\partial\Sigma} \neq 0$) and thus it is different from zero, even on-shell. According to the previous discussion, Q generates a global symmetry of the action. Two configurations which differ by a transformation generated by it represent physically different states. As a direct application of this result, we find that two flat connections A and A' whose values at the boundary differ, $(A - A')|_{\partial\Sigma} \neq 0$, cannot be connected by the action of a constraint. Thus, as we have anticipated, the values of A at the boundary represent the physically relevant degrees of freedom. The next step is to prove that there exists solutions to the equations of motion, satisfying the boundary conditions (29), with different values for A at the boundary. This is done in the next section.

By direct application of the Poisson bracket (36) one can find the algebra of two transformations with parameters η and λ not vanishing at the boundary,

$$[Q(\eta), Q(\lambda)] = Q([\eta, \lambda]) + \frac{k}{4\pi} \int_{\partial\Sigma} \eta_a d\lambda^a \tag{39}$$

where $[\eta, \lambda]^a = \epsilon^a{}_{bc}\eta^b\lambda^c$. This equation should be compared with (27). Also, note that if λ vanishes at the boundary then $Q(\lambda) = G_0(\lambda)$. One can then easily see that (39) reproduces (25) and (26) as well.

The algebra (39) provides the simplest way to determine the Poisson bracket structure on the space of functions which cannot be set to zero by the action of a constraint. We shall do this explicitly in next section.

2.6 Unitarity. An $SU(2)$ field

To end this section, we mention an important consequence of the boundary conditions (29). Namely, they provide a simple solution to one of the problems with unitarity in Chern-Simons gravity. As we have mentioned above, the gauge field $A = \omega^a + (i/l)e^a$ is complex and therefore the relevant group is $SL(2,C)$ which is non-compact. It has been argued in [30], and explicitly used for example in [31] and [25], that under some conditions in the path integral one can set $e^a = 0$ and work with the $SU(2)$ gauge field $A^a = \omega^a$. For closed manifolds, this has been shown to give a good prescription [30], but it is not the case for manifolds with a boundary.

The boundary conditions (29) lead to a simple solution to part of this problem. Indeed, expressing (29) explicitly in terms the triad and spin connection one finds,

$$\omega^a_{\bar{w}} + \frac{i}{l}e_{\bar{w}} = 0, \qquad \omega^a_w - \frac{i}{l}e_w = 0. \tag{40}$$

Using these equations, the non-zero components of A_μ and $\bar{A}_\mu$ at the boundary, namely A_w and $\bar{A}_{\bar{w}}$, can be written in terms of the spin connection as,

$$A^a_w = 2\omega^a_w, \qquad \bar{A}^a_{\bar{w}} = 2\omega^a_{\bar{w}}. \tag{41}$$

This shows that the non-zero components of the gauge field at the boundary, which in fact carry all the dynamics, are real $SU(2)$ currents.

Since all the dynamics of Chern-Simons theory will be defined at the boundary, this simple observation means that we can indeed work with two $SU(2)$ currents and forget about the non-compact nature of $SL(2,C)$. Of course in the bulk we still have an $SL(2,C)$ field and this makes the statement "all the dynamics is contained at the boundary" delicate. We shall not discussed this point anymore in this paper. Our prescription will be to treat the gauge degrees of freedom classically and to quantise the reduced phase space, after the gauge has been fixed. In the language of [3], the Chiral WZW action arises classically (by varying with respect to A_0 instead of integrating over it) and we work with its basic Poisson bracket which is the $SU(2)$ affine algebra. Actually, we shall not make explicit use of [3], but rederive the same algebras by studying global symmetries of the Chern-Simons action. Some comments on the relation between both methods will be given in Sec. 3.3.

For later convenience, we mention here that using (40) the formulae (41) can also be rewritten in terms of the triad as

$$A^a_w = \frac{2i}{l}e^a_w, \qquad \bar{A}^a_{\bar{w}} = -\frac{2i}{l}e^a_{\bar{w}}. \tag{42}$$

These formulae are more useful when constructing the metric out of the connection via $g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$.

Note finally that the equations (40) make a link between the i appearing in $A = w + (i/l)e$ and the i appearing in the complex structure given to the torus through $w = \varphi + \tau t$ (we are using here $\tau = i$).

3 The affine and anti-de Sitter solutions

As discussed in detail in the last section, the presence of a boundary means that not all the values of the field A are related by proper gauge transformations. Two solutions of the equations of motion A and A' whose values at the boundary differ by a chiral non-zero transformation are physically distinguishable solutions.

This means that the space of solutions is not trivial. In this section we shall explicitly solve the equations of motion with the boundary conditions (29) and isolate the variables which are physically relevant. We shall also find the induced Poisson bracket acting on the space of dynamical (gauge-fixed residual functions) degrees of freedom.

3.1 The affine solution

We work on the solid torus with coordinates $\{w, \bar{w}, \rho\}$ and $A = A_w dw + A_{\bar{w}} d\bar{w} + A_\rho d\rho$. Our goal in this section is to find the general solution to the equations of motion which satisfies the boundary condition (29).

The first step is to fix the gauge and eliminate the redundant degrees of freedom. We impose the gauge condition,

$$A_\rho = iJ_3, \tag{43}$$

which implies that ρ is a proper radial coordinate (see below). Our conventions for the matrices J_a are given in (18). Note that the i present in (43) means that $e_\rho \neq 0$. This is necessary because otherwise the triad would be degenerate. Next we impose the boundary condition $A_{\bar{w}} = 0$ [see (29)]. The equations of motion $F = 0$ in the coordinates $\{w, \bar{w}, \rho\}$ read explicitly,

$$\begin{aligned} \partial_\rho A_w - \partial_w A_\rho + [A_\rho, A_w] &= 0, \\ \partial_\rho A_{\bar{w}} - \partial_{\bar{w}} A_\rho + [A_\rho, A_{\bar{w}}] &= 0, \\ \partial_w A_{\bar{w}} - \partial_{\bar{w}} A_w + [A_w, A_{\bar{w}}] &= 0. \end{aligned} \tag{44}$$

The general solution to these equations in the gauge (43) and satisfying $A_{\bar{w}} = 0$ at ∂M is,

$$\begin{aligned} A_w &= b^{-1}\hat{A}(w)b, \\ A_{\bar{w}} &= 0, \\ A_\rho &= b^{-1}\partial_\rho b \end{aligned} \tag{45}$$

where $\hat{A}$ is a chiral function, $\hat{A} = \hat{A}(w)$, but otherwise arbitrary and

$$b = e^{i\rho J_3} = \begin{pmatrix} e^{-\rho/2} & 0 \\ 0 & e^{\rho/2} \end{pmatrix}. \tag{46}$$

Since $\hat{A}(w)$ is arbitrary, the space of solutions (45) is infinite dimensional. The question is whether different solutions with different values for $\hat{A}$ are related by gauge transformations or not. Consider a configuration of the form (45) and act on it with the transformation,

$$\delta A_\mu = D_\mu \eta, \qquad \eta = b^{-1}\hat{\eta}(w)b. \tag{47}$$

It is direct to see that the effect of this transformation on the solution is to produce another solution of the form (45) with $\hat{A}' = \hat{A} + \hat{D}_w\hat{\eta}$ (here $\hat{D}_w\hat{\eta}$ denotes covariant derivative in $\hat{A}$, $\hat{D}_w\hat{\eta} = \partial_w\hat{\eta} + [\hat{A}, \hat{\eta}]$). Thus, by acting on (45) with (47) we move around on the space of solutions. Actually, the transformations (47) are the most general set of transformations that leave the boundary condition and gauge fixing conditions invariant.

Since the parameter $\hat{\eta}$ appearing in (47) does not vanish at the boundary, the canonical generator of (47) is a non-zero quantity of the form (38). We then conclude that different values of the function $\hat{A}$ are connected by global transformations generated by (38) and not by the action of constraints. The function $\hat{A}$ then represents dynamical degrees of freedom. Even more, since (45) is the most general solution to the equations of motion with the boundary condition (29), the function $\hat{A}$ generates the full space of non-trivial solutions with those boundary conditions.

As it was pointed out in [25], this analysis is also valid if the boundary is located at a finite value of the radial coordinate. This observation is important, for example, if one expects to find a conformal field theory at the black hole horizon. The main difference between the metric and Chern-Simons approaches to global symmetries is that in the former one works with diffeomorphisms while in the latter with gauge transformations. Contrary to gauge transformations, diffeomorphisms are not local and move the position of the boundary. For example, the residual diffeomorphisms associated to anti-de Sitter space [5] have a non-zero radial component.

Our next step is to determine the Poisson bracket structure acting on the space of solutions, that is, the induced Poisson bracket acting on the functions $\hat{A}$. This is very easy thanks to a general theorem proved in [22]. First, we note that after the gauge is fixed (the constraint is solved and (43) is imposed) the value of Q given in (38) reduces to the boundary term,

$$\hat{Q}(\hat{\eta}) = -\frac{k}{4\pi}\int \hat{\eta}_a \hat{A}^a, \tag{48}$$

which is, as we have emphasized, different from zero. The theorem [22] states that after the gauge is fixed and one works with the induced Poisson bracket (or Dirac bracket), the charge $\hat{Q}$ satisfies the same algebra (39) as it did the full charge Q,

$$[\hat{Q}(\hat{\eta}), \hat{Q}(\hat{\lambda})]^* = \hat{Q}([\hat{\eta}, \hat{\lambda}]) + \frac{k}{4\pi}\int_{\partial\Sigma} \hat{\eta}_a d\hat{\lambda}^a \tag{49}$$

This algebra can be put in a more explicit form by defining,

$$\hat{A}^a(w) = \frac{2}{k}\sum_{n\in Z} T^a_n e^{inw}. \tag{50}$$

One finds

$$[T^a_n, T^b_m]^* = -\epsilon^{ab}{}_c T^c_{n+m} + \frac{ink}{2}\delta^{ab}\delta_{n+m,0}. \tag{51}$$

The algebra (51) is called Kac-Moody or affine algebra and represents an infinite dimensional symmetry of the space of solutions. The quantum version is obtained simply by replacing the Dirac bracket by $-i$ times the commutator,

$$[T^a_n, T^b_m] = i\epsilon^{ab}{}_c T^c_{n+m} + \frac{nk}{2}\delta^{ab}\delta_{n+m,0}. \tag{52}$$

This equation represents the algebra of the gauge-fixed basic variables (analogous to $[q,p] = i$) of Chern-Simons theory with the boundary condition (29). There are various ways to see this explicitly, which also provide alternative derivations of (52). Conceptually, the most direct derivation of this result is by starting with the three-dimensional Poisson bracket (35). Then we fix the gauge as in (43) and solve the constraint $F_{\rho\varphi} = 0$. The solutions to the constraint equation in this gauge are parametrized by the function $\hat{A}$. One can then compute the Dirac bracket of $\hat{A}$ with itself and find the affine algebra (52). Other methods yielding the same result are the WZW approach followed in [3], and the symplectic method [32]. The idea of looking at first class quantities and their algebra in Chern-Simons theory was first discussed in [28]. The presence of central terms in the algebra of global charges in Chern-Simons theory was first discussed in [33].

Note that in our applications to general relativity, the value of k is negative (see Eq. (20)). This can be cured simply by replacing[3] $T^a_n \to T^a_{-n}$, or in other words, by replacing the maximum weight condition $T^a_{-n}|0>=0$ by $T^a_n|0>=0$ $(n>0)$. In any case, we shall not consider (52) as the starting point for quantization but rather a restriction of it which yields the Virasoro algebra with central charge $c=-6k$. For this reason, in this particular calculation, it seems convenient to work with a negative k. Incidentally, note that the Virasoro algebra is also invariant under $L_n \to -L_{-n}$, $c \to -c$.

Because of its affine symmetry we shall call the solution (45) the affine solution to the equations of motion.

3.2 The anti-de Sitter solution

In principle we could consider the algebra (52) as our definition for the gauge-fixed basic quantum commutator. As we have pointed out in Sec. 2.6, the gauge field $\hat{A}$ is a real $SU(2)$ current and therefore the modes T^a_n satisfy the Hermitian condition $(T^a_n)^\dagger = T^a_{-n}$. We shall explore the quantum properties of this general metric elsewhere. Our goal here is to describe the space of solutions to general relativity with anti-de Sitter boundary conditions. The metric which follows from the general affine solution does not satisfy this requirement and thus we need to impose further restrictions on it [9].

In the conventions displayed in (18), the solution (45) for A_w can be written as,

$$A_w = \frac{i}{2} b^{-1} \begin{pmatrix} \hat{A}^3 & \hat{A}^+ \\ \hat{A}^- & -\hat{A}^3 \end{pmatrix} b \tag{53}$$

with $\hat{A}^\pm = \hat{A}^1 \pm i\hat{A}^2$ and b is defined in (46). We remind that the other components of the solution are $A_{\bar{w}} = 0$ and $A_\rho = iJ_3$, and similar expressions hold for the anti-holomorphic field.

[3] We thank M. Asorey and F. Falceto for useful conversation on this point.

The extra boundary conditions follow from looking at the form of the gauge field associated to Euclidean anti-de Sitter space which satisfies

$$\hat{A}^+ = -2, \qquad \hat{A}^3 = 0. \tag{54}$$

The anti-holomorphic field satisfies $\hat{A}^- = -2, \hat{A}^3 = 0$. These conditions were first discussed in [34], and their relation to anti-de Sitter spaces was realised in [9]. In the WZW approach, they can be incorporated in the action by adding Lagrange multipliers. This leads to a gauged WZW action whose conformal generators follow from the GKO coset construction.

We shall impose (54) as part of the boundary data. Since (45) solves the equations of motion for arbitrary values of $\hat{A}^+, \hat{A}^-$ and $\hat{A}^3$, the gauge field will still be a solution after imposing (54). This means that among the three components of the gauge field $\hat{A}^a$ only one component, $\hat{A}^-$, remains as an arbitrary function. It is convenient to rename this function as,

$$L(w) = -\frac{k}{2}\hat{A}^-(w) \tag{55}$$

and thus the solution we are interested in has the form,

$$A_w(L) = -ib^{-1}\begin{pmatrix} 0 & 1 \\ (1/k)L(w) & 0 \end{pmatrix} b, \tag{56}$$

where $L(w)$ is an arbitrary function of w, and b is given in (46). As we shall see, this solution is appropriated to anti-de Sitter spacetimes.

The boundary group leaving (54) invariant is no longer the Kac-Moody algebra (52) because that algebra does not preserve (54). Let us find the group of transformations leaving (54) invariant. First, we look for those gauge transformations $\delta A = D\lambda$ which preserve the form (56) changing only the values of L. These transformations are generated by parameters of the form [34],

$$\lambda = b^{-1}\begin{pmatrix} (i/2)\partial\varepsilon & \varepsilon \\ (1/k)\varepsilon L + (1/2)\partial^2\varepsilon & -(i/2)\partial\varepsilon \end{pmatrix} b \tag{57}$$

where $\varepsilon = \varepsilon(w)$ is an arbitrary function of w, and b is given in (46). Acting with (57) on A_w we find,

$$A_w(L) + D_w\lambda = A_w(L + \delta L), \tag{58}$$

with

$$\delta L = i(\varepsilon\partial L + 2\partial\varepsilon L + \frac{k}{2}\partial^3\varepsilon). \tag{59}$$

This shows that L is a quasi-primary field of dimension two under the residual group of transformations leaving (56) invariant.

The next step is to determine the algebra associated to these transformations. This can be done by imposing the reduction conditions (54) in the algebra (52) and computing the induced algebra. Geometrically speaking, given a Poisson bracket structure of the form $[x^a, x^b] = J^{ab}(x^a)$ (J^{ab} invertible) one defines the symplectic form σ_{ab} as the inverse of J^{ab}. The antisymmetry and Jacobi identity satisfied by J^{ab} imply that σ_{ab} is a closed 2-form. Now, let $\chi_\alpha(x^a) = 0$ a set of constraints on phase space such that $C_{\alpha\beta} := [\chi_\alpha, \chi_\beta]$ is invertible. The surface defined by $\chi_\alpha(x^a) = 0$ will be called Σ. Let σ^* the pull-back of σ into Σ. It follows that the induced Poisson bracket structure on Σ is simply the inverse of σ^* (the invertibility of σ^* is guaranteed by the invertibility of $C_{\alpha\beta}$). See, for example, [21] (chapter 2) for more details on this construction and in particular its relation with the Dirac bracket.

In terms of the modes T^a_n defined in (50), conditions (54) read

$$T^+_n = -k\delta^0_n, \qquad T^3_n = 0 \tag{60}$$

we then need to consider the matrix (evaluated on the surface (60)),

$$C := \begin{pmatrix} [T^+_n, T^+_m] & [T^+_n, T^3_m] \\ [T^3_n, T^+_m] & [T^3_n, T^3_m] \end{pmatrix} = \begin{pmatrix} 0 & k\delta_{n+m} \\ -k\delta_{n+m} & (kn/2)\delta_{n+m} \end{pmatrix} \tag{61}$$

which is indeed invertible. We remind here the form of the algebra (52) in the basis $\{T^\pm = T^1 \pm iT^2,\, T^3_n\}$,

$$\begin{aligned} [T^+_n, T^-_m] &= 2T^3_{n+m} + nk\delta_{n+m}, & (62)\\ [T^3_n, T^\pm_m] &= \pm T^\pm_{n+m}, & (63)\\ [T^3_n, T^3_m] &= \frac{kn}{2}\delta_{n+m}. & (64) \end{aligned}$$

By an straightforward application of the method explained above, the induced Poisson structure $[\ ,\ ]^*$ (or Dirac bracket) acting on the surface (54) can be written in terms of the original bracket $[\ ,\ ]$ as (sum over $n \in Z$ is assumed),

$$[a,b]^* = [a,b] + \frac{n}{2k}[a, T^+_n][T^+_{-n}, b] + \frac{1}{k}[a, T^+_n][T^3_{-n}, b] - \frac{1}{k}[a, T^3_n][T^+_{-n}, b]. \tag{65}$$

This bracket, by definition, satisfies $[a, T^3_n]^* = 0 = [a, T^+_n]^*$ for any function a, on the surface (60). Now we compute the algebra of the remaining component T^-_n. As before, it is convenient to define $L_n = -T^-_n$. The L_n's are then the Fourier modes of the function L defined in (55),

$$L(w) = \sum_{n\in Z} L_n e^{inw} \tag{66}$$

and satisfy the Virasoro algebra,

$$[L_n, L_m]^* = (n-m)L_{n+m} - \frac{k}{2} n^3 \delta_{n+m} \tag{67}$$

with a central charge

$$c = -6k. \tag{68}$$

Note that since k is negative (see (20)), the central charge in (67) is positive and thus highest weight unitary representations exist.

The form of the central term in (67) is not the standard one. One could shift L_0 in order to find the usual $n(n^2-1)$ term. However, for the black hole whose exact isometries are $SO(2) \times SO(2)$ (due to the identifications) it is more natural to leave the central term as in (67). This is also natural from the point of view of supergravity since the vacuum black hole Killing spinors are periodic [6].

The space of solutions (56) is invariant under conformal transformations generated by $L(w)$. Note that in the Chern-Simons formulation of three-dimensional gravity the Chern-Simons coupling k was related to Newton's constant G as $k = -l/4G$ (see (20)). This means that the central charge in the Virasoro algebra $c = -6k$ coincides with the Brown-Henneaux [5] central charge $c = 3l/2G$. This is not a coincidence [9]. As we shall see, the above conformal algebra represents exactly the Brown-Henneaux conformal symmetry of three-dimensional adS gravity. The relation between the reduction conditions (54) and the conformal symmetry found in [5] was stablished in [9]. A previous calculation of the central charge using the Chern-Simons formulation of three-dimensional gravity and a twisted Sugawara construction was presented in [29].

What we have done for the holomorphic sector can be repeated for the anti-holomorphic sector. The reduction conditions in this case read $\bar{T}^-_n = -k\delta^0_n$ and $\bar{T}^3_n = 0$. Since the affine $SU(2)_k$ algebra is invariant under the change $T_n^+ \leftrightarrow T_n^-$, $T^3 \to -T_n^3$, one finds the same induced algebra. The anti-de Sitter solution for the anti-holomorphic part reads

$$\bar{A}_{\bar{w}}(\bar{L}) = -ib \begin{pmatrix} 0 & (1/k)\bar{L}(\bar{w}) \\ 1 & 0 \end{pmatrix} b^{-1}, \tag{69}$$

plus $\bar{A}_w = 0$ and $\bar{A}_\rho = -iJ_3$. The residual gauge transformations are

$$\bar{\lambda} = b \begin{pmatrix} -(i/2)\partial\bar{\varepsilon} & (1/k)\bar{\varepsilon}\bar{L} + (1/2)\partial^2\bar{\epsilon} \\ \bar{\varepsilon} & (i/2)\partial\bar{\varepsilon} \end{pmatrix} b^{-1}, \tag{70}$$

where $\bar{\varepsilon} = \bar{\varepsilon}(\bar{w})$ is an arbitrary function of $\bar{w}$. Again $\bar{L}$ is a Virasoro operator and the central charge is $c = -6k$. Note that the effective coupling in the anti-holomorphic Chern-Simons theory is $-k$ because the Chern-Simons action $I[\bar{A}]$ has a minus sign in front. However, the non-zero current is now $\bar{A}_{\bar{w}}$ instead of $\bar{A}_w$. The change $k \to -k$ is compensated by $w \to \bar{w}$ and we find the same Virasoro algebra with the same central charge in both cases.

We shall call the gauge fields (56) and (69) the anti-de Sitter solution of the equations of motion because they are appropriated to anti-de Sitter spacetimes.

3.3 The WZW and Liouville actions

We have displayed in this section two solutions to the Chern-Simons equations of motion with an affine (Sec. 3.1) and Virasoro (Sec. 3.2) symmetries. We have also argued that, at least classically, the corresponding algebras can be interpreted as basic Poisson brackets acting on the corresponding phase spaces. A natural and powerful way to justify this point is by studying the induced theories at the boundary for the corresponding boundary conditions. It is known [3] that under the boundary condition (29), the Chern-Simons action reduces to a WZW action at the boundary whose basic Poisson bracket, first calculated in [35], is the affine algebra (52). Reducing the affine solution via (54) then gives (67). At this point, a natural question to ask is what is the boundary action (analogue to the WZW action) which would give rise directly to the Virasoro algebra (67) as its basic Poisson bracket. We do not know the answer to this question. However, an alternative route can be taken. It was shown in [9] that the two chiral WZW actions arising in Chern-Simons gravity can be combined into a single non-Chiral action via $g = g_1^{-1}g_2$. Furthermore, the reduction conditions (54) applied to the non-Chiral theory lead to a Liouville action [36] which has the expected conformal symmetry with a central charge equal to $c = -6k$. The solutions of three-dimensional gravity can be classified in terms of the solutions of Liouville theory. One finds that the different monodromy conditions (elliptic, parabolic and hyperbolic) led to the three classes of solutions: conical singularities, extreme, and black holes[37]. See [8] for a direct relation between the Liouville energy momentum tensor and the anti-de Sitter boundary conditions, without using the Chern-Simons formalism. The Liouville field has also appeared in [38] in the study of solutions to the 2+1 classical equations with a positive cosmological constant.

The Liouville action is certainly a good candidate to describe the dynamics of 2+1 gravity with anti-de Sitter boundary conditions. However, it should be kept in mind that its derivation from the WZW model is not unique and a better control on some global issues is necessary. First, merging the two Chiral WZW actions into a single one through $g = g_1^{-1} g_2$ is not unique because g is invariant under $g_1 \to A g_1$, $g_2 \to A g_2$ with A arbitrary. Second, the Liouville action arises in terms of a Gauss decomposition of the group element which is not global. For these reasons we have not committed ourselves to any particular form for the boundary action but instead we have treated the basic Poisson bracket algebras as the starting point for quantization.

4 A quantum spacetime

We have found in the last section a general solution for the classical equations of motion with prescribed boundary conditions. We have also found the induced Poisson bracket structure acting on the gauge-fixed dynamical functions. Our aim in this section is to quantise those spaces and apply the results to three-dimensional gravity.

However, an important warning is necessary here. The space of solutions that we have displayed (affine and anti-de Sitter solutions) are explicitly not coordinate invariant. This means that we could have chosen other coordinates to describe that space and it is not guaranteed that the corresponding quantum versions would be equivalent. For example, we know that a full quantization of the Chern-Simons action with compact groups induces a shift in the coupling constant [2, 3, 4] which is not seen in our gauge-fixed approach (although it is suggested by the Sugawara form of the conformal generators; see [29] for a construction of the Virasoro generators using a twisted Sugawara operator). We shall not attack this problem here in the belief that at least in the large k limit our results can be trusted.

It is worth stressing here that the fact that we have found non-Abelian Poisson structures (Virasoro and Kac-Moody algebras) is a consequence of the self-interacting character of gravity. The non-Abelian pieces in those algebras are measured by the coupling k which in the semiclassical limit, when gravity becomes linearized, goes to infinity.

4.1 The metric

The general solution to Einstein equations in three dimensions with anti-de Sitter boundary conditions can be found using the results of last section together with the correspondence between metrics $g_{\mu\nu}$ and connections $A^a_\mu, \bar{A}^b_\nu$ discovered in [1]. Given the connections $A^a_\mu, \bar{A}^a_\mu$, one constructs the Lorentz vector $e^a_\mu = (l/2i)(A^a_\mu - \bar{A}^a_\mu)$ and then the metric $g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$. It follows from the analysis of [1] that if A^a_μ and $\bar{A}^a_\mu$ satisfy the Chern-Simons equations of motion, then $g_{\mu\nu}$ satisfies the three-dimensional gravitational equations.

For our calculations it will be useful to define the matrix,

$$e_\mu = \frac{l}{2i}(A_\mu - \bar{A}_\mu) \tag{71}$$

where $A = A^a J_a$ and $\bar{A} = \bar{A}^a J_a$. The spacetime metric is then given by

$$g_{\mu\nu} = -2\mathrm{Tr}(e_\mu e_\nu). \tag{72}$$

The conventions for the J_a matrices are displayed in (18).

The affine solution (45) (and its anti-holomorphic part) is certainly a very general and interesting solution for the Chern-Simons equations of motion, however, the induced metric does not satisfy the anti-de Sitter boundary conditions prescribed in [5]. This is the reason that we have also considered the reduction of (45) via (54) which leaves an asymptotically anti-de Sitter spacetime, with a conformal symmetry.

Consider the anti-de Sitter solutions (56) and (69) for the Chern-Simons equations of motion and let us compute the associated metric. Using (71) we compute the components of the triad,

$$e_w \;\; = \;\; -\frac{l}{2}\begin{pmatrix} 0 & e^\rho \\ e^{-\rho}L/k & 0 \end{pmatrix}, \tag{73}$$

$$e_{\bar{w}} = \frac{l}{2}\begin{pmatrix} 0 & e^{-\rho}\bar{L}/k \\ e^{\rho} & 0 \end{pmatrix}, \tag{74}$$

$$e_{\rho} = lJ_3, \tag{75}$$

and then using (72) we find the metric,

$$ds^2 = 4Gl(Ldw^2 + \bar{L}d\bar{w}^2) + \left(l^2 e^{2\rho} + 16G^2 L\bar{L}e^{-2\rho}\right) dw d\bar{w} + l^2 d\rho^2. \tag{76}$$

Here we have used the value of k given in (20). This is the metric that was displayed in Sec. 1.1. We can now go through the properties listed in that section and check their validity.

First, by construction (76) solves the Einstein equations because the corresponding gauge fields solve the Chern-Simons equations. This, of course, can be proved explicitly by checking that (76) has constant curvature. Since $L(w)$ and $\bar{L}(\bar{w})$ are arbitrary functions, the metric (76) provides an infinite number of solutions to Einstein equations with a negative cosmological constant in three dimensions. These solutions represent different physical states because two metrics with different values of $L, \bar{L}$ are related by a global diffeomorphism (global diffeomorphism here is what was called "improper" in [23]). At this point it is necessary to prove that the notion of global diffeomorphism has an analogue within general relativity. Since (76) has constant curvature, there exists a change of coordinates mapping (76) into the anti-de Sitter metric (11). The question is whether that change of coordinates is generated by one of the constraints of general relativity or not.

In complete analogy with the gauge case, we define global diffeomorphisms as coordinate transformations which are not generated by the constraints of general relativity[20]. Rather, global diffeomorphisms are generated by non-zero quantities which enter as boundary terms in their canonical generators. This has been analysed in detail in [5] from where we conclude that two metrics of the form (76), which only differ of the values of L and $\bar{L}$, are connected by a global diffeomorphism. The functions L and $\bar{L}$ then represent physical degrees of freedom from the gravitational point of view.

4.2 Diffeomorphisms in Chern-Simons gravity

It is interesting and instructive to prove explicitly that there exists a change of coordinates $\{w, \bar{w}, \rho\} \to \{w', \bar{w}', \rho'\}$ which preserve the form of (76) changing only the values of L and $\bar{L}$. This is the goal of this paragraph. To find these transformations we can either do it by brute force acting with Lie derivatives on (76), or by using the results of the last sections making a dictionary between gauge transformations and diffeomorphisms. We shall follow this last procedure.

It is well known that, due to the flatness of the gauge field, in Chern-Simons theory the diffeomorphism invariance is not an independent symmetry. Indeed, a diffeomorphism along a vector field ξ^μ can be written as a gauge transformation with a parameter $\lambda^a = A^a_\mu \xi^\mu$ [14]. The converse is, in general, not true. However, in the case of Chern-Simons gravity where the relevant group is $SL(2,C)$ and an invertible triad exists, one can prove that all gauge transformations act on the metric via diffeomorphisms.

More explicitly, let $g_{\mu\nu}$ the metric associated to a particular configuration $A^a_\mu, \bar{A}^b_\nu$ through (71) and (72). Now, consider an arbitrary gauge transformation with parameters $\lambda^a, \bar{\lambda}^a$ acting on $A^a, \bar{A}^a$. It follows that the transformed metric (associated to the transformed fields) is related to the original one by a diffeomorphism generated by a vector field $\xi^\mu_{(\lambda,\bar{\lambda})}$.

To prove this statement, and find the explicit formula for $\xi^\mu_{(\lambda,\bar{\lambda})}$, consider the action of the gauge group on the triad. From (71) we find that under a gauge transformation $\delta A_\mu = D_\mu \lambda$, $\delta \bar{A}_\mu = \bar{D}_\mu \bar{\lambda}$ the triad changes according to,

$$\delta e_\mu = \frac{l}{2i} D^{(w)}_\mu (\lambda - \bar{\lambda}) - \frac{1}{2}[e_\mu, \lambda + \bar{\lambda}]. \tag{77}$$

Here we have used that $A_\mu + \bar{A}_\mu = 2\omega_\mu$ where $\omega_\mu = \omega^a_\mu J_a$ is the spin connection, and $D^{(w)}_\mu$ denotes its associated covariant derivative. The second term in (77) is a Lorentz rotation of the triad which does not change the metric. We then concentrate on the first term. Let us define the $SO(3)$ vector ρ^a and its associated vector field $\xi^\mu_{(\rho)}$ by,

$$\rho^a = \frac{l}{2i}(\lambda^a - \bar{\lambda}^a), \tag{78}$$

$$\xi^\mu = e^\mu_a \rho^a. \tag{79}$$

We assume that e^a_μ is invertible then (79) does make sense. We now study how does the transformation,

$$\delta e^a_\mu = D^{(w)}_\mu \rho^a = D^{(w)}_\mu (e^a_\nu \xi^\nu), \tag{80}$$

change the metric. Define the Christoffel symbols in the standard way by

$D^{(w)}_\mu e^a_\nu = \Gamma^\sigma_{\nu\mu} e^a_\sigma$ [4]. The transformation (80) becomes $\delta e^a_\mu = e^a_\nu \xi^\nu{}_{;\mu}$ where the semicolon denotes standard covariant derivative. The action of this transformation on the metric is,

$$
\begin{aligned}
\delta g_{\mu\nu} &= \delta e^a_\mu e_{a\nu} + e^a_\mu \delta e_{a\nu}, \\
&= (e^a_\sigma \xi^\sigma{}_{;\mu}) e_{a\nu} + e^a_\mu (e_{a\sigma} \xi^\sigma{}_{;\nu}), \\
&= \xi_{\mu;\nu} + \xi_{\nu;\mu},
\end{aligned}
\tag{81}
$$

where in the last line we have used the definition of the metric tensor and the identity $g_{\mu\nu;\sigma} = 0$. Thus, a transformation in the triad of the form (80) is indeed seen in the metric as a diffeomorphism. Since the gauge transformations acting on $A, \bar{A}$ produce (up to a Lorentz rotation) a transformation of the form (80) with ρ^a given in (78), we conclude that the gauge group acts on the metric via a diffeomorphism with a parameter defined in (79).

Now we apply this result to the particular case of the residual gauge transformations (57) and (70). The residual vector field $\xi^\mu_{(\varepsilon,\bar\varepsilon)}$ associated to those transformations is computed directly from (57) and (70) plus (78) and (79). The formulae for the triad are given in (73)-(75).

It should be clear from the above analysis that $\xi^\mu_{\varepsilon,\bar\varepsilon}$ generates a residual symmetry of the metric (76). This can be summarised in the following table. $A_\mu(L)$ and $\bar{A}^a_\nu(\bar{L})$ represent the residual connections (56) and (69), and $g_{\mu\nu}(L,\bar{L})$ the associated metric (76). Under the residual gauge transformations (57) and (70) the gauge field and metric transform according to:

$$
\boxed{
\begin{array}{ccccc}
\begin{array}{c} A_\mu(L) \\ \bar{A}_\nu(\bar{L}) \end{array} & \rightarrow & \begin{array}{c} A_\mu(L) + \delta A_\mu \\ \bar{A}_\nu(\bar{L}) + \delta \bar{A}_\nu \end{array} & \begin{array}{c} = \\ = \end{array} & \begin{array}{c} A_\mu(L+\delta L) \\ \bar{A}_\nu(\bar{L}+\delta\bar{L}) \end{array} \\
\Downarrow & & \Downarrow & & \Downarrow \\
g_{\mu\nu}(L,\bar{L}) & \rightarrow & g_{\mu\nu}(L,\bar{L}) + \mathcal{L}_{\xi_{(\varepsilon,\bar\varepsilon)}} g_{\mu\nu} & = & g_{\mu\nu}(L+\delta L, \bar{L}+\delta\bar{L})
\end{array}
}
$$

where δL is given in (59), and the same expression holds for $\delta\bar{L}$. The equalities in the first two lines simply express the fact that the residual gauge transformations leave the gauge field invariant changing only the values of L and $\bar{L}$. The third line contains non-trivial information. First, as we discussed

[4]The meaning of this definition can be uncovered by writing it in the form $\Gamma^\sigma_{\mu\nu} = e^\sigma_a \omega^a{}_{b\nu} e^b_\mu + e^\sigma_a e^a_{\mu,\nu}$. This is the transformation law of a connection, $\omega^a{}_{b\nu} \rightarrow \Gamma^\sigma{}_{\mu\nu}$, under the change of basis from the coordinate basis ∂_μ to the orthonormal frame $\vec{v}_a$ described by the matrix e^a_μ: $\partial_\mu = e^a_\mu \vec{v}_a$.

above, the metric associated to the transformed gauge fields is related to the original metric via a diffeomorphism parametrised with the residual vector field $\xi^\mu_{(\varepsilon,\bar\varepsilon)}$. Then, since the transformed metric can also be written in terms of $A(L+\delta L), \bar A(\bar L+\delta\bar L)$, we find the last equality and conclude that the vector field $\xi^\mu_{(\varepsilon,\bar\varepsilon)}$ generates a residual diffeomorphism of the metric (76). This analysis exhibit the power of the Chern-Simons formalism. To discover that the metric (76) has a residual conformal invariance –not only asymptotically– using Lie derivatives would have been extremely complicated.

Let us work out explicitly the case on which $\bar L = 0$. The metric (76) reduces to the simple form

$$ds^2 = 4GlL(w)dw^2 + l^2 e^{2\rho} dw d\bar w + l^2 d\rho^2. \tag{82}$$

We act on this metric with the holomorphic residual transformation generated by (57). The associated residual vector $\xi^\mu = \delta x^\mu$ is computed from (57) and the triad (73)-(75) with $\bar L = 0$. In the coordinates $\{w, \bar w, \rho\}$ it reads,

$$\delta\rho = -\frac{i}{2}\partial\varepsilon, \qquad \delta w = i\varepsilon, \qquad \delta\bar w = -\frac{i}{2}e^{-2\rho}\partial^2\varepsilon. \tag{83}$$

Transforming the metric (82) with this vector one finds the same metric with L replaced by $L' = L + \delta L$, and

$$\delta L = i(\varepsilon\partial L + 2\partial\varepsilon L - \frac{l}{8G}\partial^3\varepsilon). \tag{84}$$

Since $l/8G = c/12 = -k/2$ with c and k given respectively in (3) and (20), we find consistency with (59), as expected. As a further check, we can now transform (82) with the anti-holomorphic residual transformation generated by (70). Since $\bar L$ is a quasi-primary field, we expect that this transformation will not preserve $\bar L = 0$ and thus the metric (82) will be transformed into (76) with $\bar L = \delta\bar L$. Indeed, from (70) and (73)-(75) we find the associated transformation,

$$\delta\rho = -\frac{i}{2}\bar\partial\bar\varepsilon, \qquad \delta w = -\frac{i}{2}e^{-2\rho}\bar\partial^2\bar\varepsilon, \qquad \delta\bar w = i\varepsilon + \frac{2iGL}{l}e^{-4\rho}\bar\partial^2\bar\varepsilon. \tag{85}$$

We act on (82) with this vector and find a metric of the form (76) with $\bar L = (-il/8G)\bar\partial^3\bar\varepsilon$. This is exactly the right transformation, in accordance with (84) applied to the anti-holomorphic field $\bar L(\bar w)$.

As a last example of the conformal residual symmetries of (76) we mention here the case of the finite (holomorphic) exponential map. To simplify the notation we set here $4G = 1$ and $l = 1$. Let us make the finite change of coordinates on (82) $\{w, \bar{w}, \rho\} \to \{z, \bar{w}', \rho'\}$ defined by

$$z = e^{-iw}, \quad \bar{w}' = \bar{w} + (i/2)e^{-2\rho}, \quad e^{2\rho'} = ie^{2\rho+iw} \tag{86}$$

This transformation maps the cylinder w into the plane z, but leaves $\bar{w}$ in the cylinder. Note that here we are using explicitly the independence of w and $\bar{w}$. We act on the metric (82) with this change of coordinates and find again the same metric with $L(w)$ replaced by $T(z) = -z^{-2}(L(w)+1/2)$. The shift $1/2 = 6/12$, of course, corresponds to the Schwarztian derivative of the map (86). (In the conventions $4G = 1$ and $l = 1$ the central charge (3) is equal to 6.)

4.3 Black holes

As we have mentioned above, the metric (76) reduces to a three-dimensional black hole [11, 12] when $L = L_0$ and $\bar{L} = \bar{L}_0$ are constants. This can be proved as follows. First, we define the constants M,J and $r_\pm$ by

$$L_0 + \bar{L}_0 \quad = Ml = \frac{r_+^2 + r_-^2}{8Gl}, \tag{87}$$

$$L_0 - \bar{L}_0 \quad = J = \frac{2r_+ r_-}{8Gl}. \tag{88}$$

Next we define the real coordinates φ, t and r by,

$$w = \varphi + it, \tag{89}$$

and

$$r^2 = r_+^2 \cosh^2(\rho - \rho_0) - r_-^2 \sinh^2(\rho - \rho_0). \tag{90}$$

The constant ρ_0 is given by $e^{2\rho_0} = (r_+^2 - r_-^2)/(4l^2)$. This radial definition has the property $l^2 d\rho^2 = N^{-2} dr^2$ where N^2 is the lapse function appearing in the black hole metric (8). The constants $r_\pm$ are the solutions of the equation $N^2(r_\pm) = 0$. After a long but direct calculation one can prove that the metric (76), in the coordinates $\{t, r, \varphi\}$, is exactly equal to the metric (8) with mass M and angular momentum J.

Since we are working in the Euclidean sector, the coordinate t appearing in (89) and (8) is periodic, $0 \leq t < \beta$, with $\beta = 2\pi l^2 r_+/(r_+^2 - r_-^2)$. In order to fix the period of the time coordinate to be independent of the black hole parameters (and thus fix the complex structure of the torus), one can define $z = \varphi + \tau x^0$ with $0 \leq x^0 < 2\pi$ and $\tau = i\beta/2\pi$. Since φ and x^0 are periodic, the complex coordinate z is defined on a torus,

$$z \sim z + 2\pi n + 2\pi\tau m, \qquad n, m \in Z. \tag{91}$$

with τ its modular parameter. Introducing τ is particularly convenient when studying modular invariance on the black hole manifold[25, 39]. We shall not deal with this issue here, so we use (89).

4.4 The quantum space of metrics. State counting

We have described in the last paragraph a set of metrics parametrized by two functions whose induced Poisson bracket yield the Virasoro algebra with a non-zero central charge. We shall now promote the algebra (67) to be a quantum algebra and study the properties of the associated quantum metric.

The unitary representations of the Virasoro algebra for a given positive central charge c are parametrized by a single real positive number h. In the semiclassical limit with $-k$ large, the Virasoro central charge $c = -6k$ is then large. Under these conditions, there exists one unitary representation for each conformal dimension h. We start with the vacuum state $|h>$ satisfying $L_0|h>= h|h>$ and $L_n|h>= 0$ $(n > 0)$. The excited states are constructed with the negative modes L_{-n} acting on $|h>$. The full representation, for a given h, is spanned by the vectors $|n_1, ..., n_r; h>:= L_{-n_1} \cdots L_{-n_r}|h>$ with $r = 1, 2, ...$ The same construction has to be repeated for the other Virasoro algebra $\bar{L}_n$.

In standard conformal field theory, the values of h are not arbitrary. They are equal to the conformal dimensions of the primary fields ϕ_h of the theory. The state $|h>$ is created by ϕ_h via $|h>= \phi_h(0)|0>$ where $|0>$ is the true conformal vacuum. In our situation, we do not have a field theory at the boundary (of course it could be Liouville theory, see Sec. 3.3 for a discussion on this point) but only the Virasoro algebra. The usual state-operator map will then be missing until we decide which is right conformal field theory.

Once we promote the modes L_n and $\bar{L}_m$ to be operators acting on Fock space, the metric (76) becomes an operator that we shall denote by $d\hat{s}^2$. Note that since the metric (76) is an algebraic function in L and $\bar{L}$ which does not involve products of non-commuting operators, $d\hat{s}^2$ is well defined in the operator sense.

We can now define a natural map from Fock space to the space of classical solutions. For each state $|\Psi>$ in Fock space, we associate a classical solution to the equations of motion given by,

$$ds^2_\psi =< \psi|d\hat{s}^2|\psi > \tag{92}$$

where ds^2_ψ is the metric (76) with $L =< \psi|\hat{L}|\psi >$ and $\bar{L} =< \psi|\hat{\bar{L}}|\psi >$. Since (76) is a solution for arbitrary values of L and $\bar{L}$, the metric (92) is a solution for any state $|\psi >$. More interesting, the full set of solutions (76) can be generated by the above map.

According to (92), every state $|\psi >$ induces a unique classical solution. The converse is not true. For a given classical solution there may be many associated states. In particular, there are many states associated to a given black hole of mass M and angular momentum J. As explained before, the metric (76) gives rise to a black hole when L and $\bar{L}$ are constants and related to M and J by (87) and (88). Let $|M, J; \lambda >$ the set of states in the Hilbert space such that they satisfy

$$\begin{aligned} < M, J; \lambda|(L_n + \bar{L}_n)|M, J; \lambda > &= lM\, \delta^0_n, \\ < M, J; \lambda|(L_n - \bar{L}_n)|M, J; \lambda > &= J\, \delta^0_n, \end{aligned} \tag{93}$$

for all $\lambda = 1, 2, 3, ..., \rho(M, J)$. These states generate through (92) a black hole of mass M and angular momentum J. We can then formulate the problem of black hole degeneracy as whether the logarithm of the number of these states, $\ln \rho(M, J)$, is equal to the Bekenstein-Hawking entropy of the corresponding black hole of mass M and angular momentum J or not. The answer to this question depends on the structure of the Hilbert space.

Let us first work under the assumption that the Virasoro algebra is the basic quantum commutator of the theory. In this case, the counting is very simple. The states $|n_1, ..., n_r; h >$, properly normalized, precisely have the property,

$$< n_1, ..., n_r; h|L_n|n_1, ..., n_r; h >= L_0 \delta^0_n \quad \text{with} \quad L_0 = h + \sum_{i=1}^{r} n_i. \tag{94}$$

The number of these states, $\rho(L_0, \bar{L}_0)$, is then equal to number of ways that one can write an integer as a sum of integers. For large values of L_0 and $\bar{L}_0$ this number is approximated by the well-known Ramanujan formula,

$$\rho_{c'}(L_0, \bar{L}_0) = e^{2\pi\sqrt{c' L_0/6} + 2\pi\sqrt{c' \bar{L}_0/6}}, \tag{95}$$

with $c' = 1$. Unfortunately, this naive counting does not give the right result. Inserting $lM = L_0 + \bar{L}_0$ and $J = L_0 - \bar{L}_0$ in (95) gives an entropy equal to $S = c^{-1/2} A/4G$, where c is the central charge (3) and $A = 2\pi r_+$ is the perimeter of the horizon. (The relation between the different parameters is given in (87) and (88).) The prefactor $c^{-1/2}$ shows that our naive procedure is not yet correct because we would expect the degeneracy of states to be equal to the Bekenstein-Hawking value.

If we do not regard (67) as the basic algebra but only as representing the symmetry algebra of some underlying conformal field theory, then an elegant and striking way to relate (67) with the correct Bekenstein-Hawking entropy is available [7]. Suppose that the algebra (67) represents the Virasoro algebra associated to some conformal field theory with central charge c. Suppose also that this CFT is unitary, in the sense that $L_0, \bar{L}_0 > -c/24$ (note that we are using the Virasoro generators which vanish for the vacuum black hole), and that the partition function,

$$Z[\tau] = \mathrm{Tr}\, e^{2\pi i \tau L_0 - 2\pi i \bar{\tau} \bar{L}_0}, \tag{96}$$

is modular invariant. This means

$$Z[\tau'] = Z[\tau], \qquad \tau' = \frac{a\tau + b}{c\tau + d}, \tag{97}$$

for any $a, b, c, d \in Z$ and $ad - bc = 1$. Then, it follows [40] that that number of states with L_0 and $\bar{L}_0$ fixed is again given by (95) but this time with $c' = c$. The associated entropy is then exactly equal to the Bekenstein-Hawking value $S = A/4G$ with $A = 2\pi r_+$ equal to the perimeter of the horizon [7].

As stressed in [10], this result is too beatiful to be wrong. Even more, recently [41, 42], it has been shown that under some boundary conditions at the horizon, similar results can be applied to higher dimensions. These are exiting results which bring closer the long standing dream of a statistical mechanical description for the Bekenstein-Hawking entropy. However, there remains to find the conformal field theory responsible for the degrees of freedom and, most importantly, to determine whether general relativity is enough to describe that CFT, or other degrees of freedom like string theory are necessary.

An alternative route to get the right counting was suggested in [43]. For integer values of the central charge c, there is a natural way to add degrees of freedom to the theory in such a way that the counting yields the right result.

The idea is that the Virasoro algebra (67) can be regarded as a sub-algebra of another Virasoro algebra, with central charge 1 and generators Q_n, via the formula,

$$L_n = \frac{1}{c} Q_{cn}. \tag{98}$$

See [44] and references therein for a detailed description of this embedding. (The formula (98) has also appeared in [45].) The number of states associated to the representations of the operators Q_n is again given by (95) with $c' = 1$ and L_0 replaced by Q_0. Since by (98) $Q_0 = cL_0$, this yields the right result when using (87) and (88). The main problem with this approach is that we do not know how to relate the gravitational degrees of freedom to the generators Q_n. Perhaps one should look for other boundary conditions, generalising (54), which may give other conformal structures, generalising (67). This issue is presently under investigation.

Whether the Virasoro operators are fundamental variables or not, this will not change our quantum geometry picture. The microscopical origin of the black hole degeneracy is associated to different states (living in the correct CFT) which generate the same classical metric through (92).

5 Final remarks

Maldacena [46] has conjectured a duality between large N super-conformal field theory in four dimensions and Type IIB string theory compactified on adS$_5 \times S_5$. This relation has become known as adS/CFT correspondence due to the relation between the symmetry groups in each theory. The result of Brown and Henneaux [5] relating adS$_3$ and a conformal algebra in 1+1 dimensions can also be regarded as an adS/CFT correspondence. Note however that contrary to the higher dimensional case, this relation involves only asymptotic adS space whose isometry group is infinite dimensional. In [37, 47, 48] the relation between these two aspects of the adS/CFT correspondence has been explored.

Finally, we would like to mention here a surprising motivation to study three-dimensional gravity. It has been shown in [49] and [50] (see also the recent review [51]) that there exists duality transformations relating five-dimensional black holes with three-dimensional ones. This means that everything we can learn about three-dimensional quantum gravity can be useful to higher dimensional situations.

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