Hamiltonian Formalism, Regge-Teitelboim charges and some AdS/CFT applications

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Disclaimer

- These notes have not been proof read for signs, factors $\frac{1}{2}$, π,etc.
- The content is based on the work of many people. I apologize for not quoting them appropriately. Un updated version will contain proper references.
- The author has not contributed to this subject. (But errors in these notes are of course his responsibility.)
- I would like to take this opportunity to thank Claudio Bunster and Marc Henneaux from whom I have learned everything I know in this subject.

Contents

1. Hamiltonian Formalism for gauge theories.
   - Examples.
   - We shall not discuss the Dirac algorithm!

2. Boundary terms, boundary conditions. Asymptotic symmetries, charges.
   - The asymptotic transmutation of gauge symmetries into Noether global symmetries

3. The magic of the Cardy formula. The central charge and black hole entropy.
   - Asymptotic centrally extended Virasoro algebra

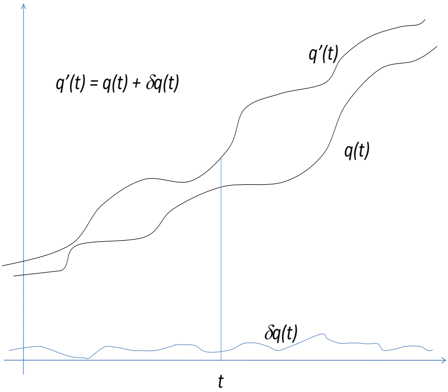
The Hamiltonian formulation of mechanics

$$I[p,q] = \int dt \left[p_i \dot{q}^i - H(p,q)\right] \qquad \left[\begin{array}{l} \dot{q}^i = \frac{\partial H}{\partial p_i} = [q^i, H] \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} = [p_i, H] \end{array}\right]$$

has an interesting and powerful structure:

1. Symplectic geometry, Hamilton-Jacobi theory,...
2. The first (general) quantization method
3. The Energy $E = H(p,q)$ functional is built-in in the formalism.
4. Gauge theories have a clear structure in Hamiltonian form: number of degrees of freedom; splitting between dynamical and non-dynamical fields; conserved charges;...

Noether theorem in one slide: Let $\delta q^i(t), \delta p_j(t)$ be small functions. Given $q(t)$ we build a new function $q'^i(t) = q^i(t) + \delta q^i(t)$.



δq is a symmetry if, *for all* $q(t)$,

$$\delta I[q] = I[q'] - I[q] = \int dt \frac{dB}{dt}$$

for some local $B(q,p)$. On the other hand, the on-shell variation is always a boundary term:

$$\delta I[q] = \int dt \frac{d(p_i \delta q^i)}{dt}$$

Subtracting both variations we derive the Noether conservation law:

$$\frac{d}{dt}(B - p_i \delta q^i) = 0 \quad \Rightarrow \quad \epsilon Q \equiv B - p_i \delta q^i = \text{conserved}$$

The converse theorem is also true. Let $Q(p,q)$ be a conserve charge, i.e., a function on phase space that commutes with the Hamiltonian,

$$[Q,H]=0. \qquad \text{(Poisson brackets)}$$

Then, the following transformations Q are a symmetry of the action:

$$\begin{aligned} \delta q^i &= [q^i, Q] = \frac{\partial Q}{\partial p_i} \\ \delta p_i &= [p_i, Q] = -\frac{\partial Q}{\partial q^i} \end{aligned}$$

Proof: First, for any function $A(q,p)$ of the canonical variables it follows: $\delta A = \frac{\partial A}{\partial q^i}\delta q^i + \frac{\partial A}{\partial p_i}\delta p_i = [A,Q]$. In particular the Hamiltonian is invariant,

$$\delta H = [H,Q] = 0.$$

The variation of the Kinetic term:

$$\begin{aligned} \delta(p_i\dot{q}^i) = \delta p_i \dot{q}^i - \dot{p}_i\delta q^i + \frac{d}{dt}(p_i\delta q^i) &= -\frac{\partial Q}{\partial q^i}\dot{q}^i - \dot{p}_i\frac{\partial Q}{\partial p_i} + \frac{d}{dt}(p_i\delta q^i) \\ &= -\frac{dQ}{dt} + \frac{d}{dt}(p_i\delta q^i) \end{aligned}$$

Noether Symmetries

$$\frac{dQ(p,q)}{dt} = 0 \qquad \Leftrightarrow \qquad \begin{array}{l} \delta q^i = [q^i, Q] \\ \delta p_i = [p_i, Q] \end{array}$$

Gauge Symmetries

$$\phi(q,p) = 0 \qquad \Leftrightarrow \qquad \begin{array}{l} \delta q^i = [q^i, \phi] \\ \delta p_i = [p_i, \phi] \end{array}$$

We shall study this structure for gauge theories. In particular how, in field theory, gauge symmetries get transmuted into Noether symmetries asymptotically.

Before proceeding... A note on constraints generating symmetries: It may seem strange that a zero quantity can generate a non-zero transformation.

A free relativistic particle satisfies the constraint:

$$\phi = p_\mu p^\mu + m^2 = 0$$

The coordinates and momenta have canonical poisson brakets:

$$[X^\mu, p_\nu] = \delta^\mu_{\ \nu}.$$

Then

$$\begin{aligned} \delta X^\mu &= [X^\mu, p^\alpha p_\alpha + m^2] \\ &= 2p^\mu \neq 0 \end{aligned}$$

So, the constraint could be zero and yet its Poisson brackets different from zero.

Overview of these Lecture. The oldest and best well-known gauge theory is Maxwell electrodynamics. Let us walk through its gauge symmetries, constraints, Lagrange multipliers and conserved charges.

Maxwell theory is gauge invariant ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$):

$$I[A] = -\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu}, \qquad \delta A_\mu = \partial_\mu \epsilon(x)$$

[The parameter $\epsilon(x)$ is an arbitrary function of space and time]

What is the impact of the gauge symmetry on the dynamics of this field?

Let us explore Maxwell theory in Hamiltonian form:

Foliate spacetime into space hypersurfaces with local coordinates x^i, and $x^0 = t$:

$$A_\mu = (A_0, A_i) \quad \Rightarrow \quad \delta A_0 = \dot{\epsilon}, \quad \delta A_i = \partial_i \epsilon$$

Now expand the action,

$$
\begin{aligned}
I[A_0, A_i] &= -\frac{1}{4}\int dt \int d^3x \left[2F^{0i}F_{0i} + F^{ij}F_{ij}\right] \\
&= \int dt \int d^3x \left[\frac{1}{2}\dot{A}^i\dot{A}_i - \dot{A}^i\partial_i A_0 + \frac{1}{2}\partial_i A_0 \partial^i A_0 - \frac{1}{4}F^{ij}F_{ij}\right]
\end{aligned}
$$

A_0 enters with no-derivatives. We do not introduce a momenta for it.

$$
p_i = \frac{\partial L}{\partial \dot{A}^i} = \dot{A}_i - \partial_i A_0 = E_i \qquad \Rightarrow \qquad \dot{A}_i = E_i + \partial_i A_0
$$

The Hamiltonian (density) is:

$$
\begin{aligned}
\mathcal{H}(E, A) &= E_i\dot{A}^i - L \\
&= \frac{1}{2}E_iE^i + F^{ij}F_{ij} - A_0\partial_i E^i \\
&= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) - A_0 \nabla\cdot\vec{E}
\end{aligned}
$$

where $B_i = \epsilon_{ijk}F^{jk}$.

The Hamiltonian action becomes

$$I[A_i, E^j, A_0] = \int dt \int d^3x \left[E^i \dot{A}_i - \underbrace{\frac{1}{2}(\vec{E}^2 + \vec{B}^2)}_{\text{Hamiltonian}\neq 0} + A_0 \nabla \cdot \vec{E} \right]$$

- The energy density of an electromagnetic field $\vec{E}, \vec{B}$ is

$$\rho = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

- A_0 is a Lagrange multiplier. Its variation implies the constraint $\nabla \cdot \vec{E} = 0$ (Gauss law in vacuum), but gives no information for its time evolution. $A_0(x, t)$ is not determined by any equation.
  - This theory seems to be incomplete. One of its functions is not determined by the field equations.

Let us look at the Hamiltonian equations of motion. Varying with respect to A_i and E^i we obtain two first order equations:

$$\begin{aligned} \dot{\vec{A}} &= \vec{E} + \nabla A_0 \\ \dot{\vec{E}} &= \nabla \times \vec{B} \end{aligned}$$

The arbitrariness, ∇A_0, shows up in the evolution of $\vec{A}$, in the form of a *gauge transformation*:

$$\vec{A}(t_0 + \delta t) = \vec{A}(t_0) + \vec{E}\delta t + \underbrace{\nabla(\delta t A_0)}_{\text{gauge}}$$

Given initial conditions $\vec{A}(t_0)$, the field value at $t_0 + \delta t$ is determined up a gauge transformation with a parameter $\delta t\, A_0$. The evolution tell us that $\vec{A}$ and $\vec{A} + \nabla\lambda$ must represent the same physical situation.

Of course we knew this already for electrodynamics. But there are cases where the gauge symmetry is not that obvious/well-known.

Let us look at the variation of the action more closely.

$$\delta I = \int dt \int d^3x \left[\delta\vec{E}\cdot(\dot{\vec{A}} - \vec{E}) + A_0\,\nabla\cdot\delta\vec{E}\right] \qquad (E^r \simeq \frac{Q}{r^2} + \frac{d}{r^3} + \cdots)$$

$$= \int dt \int d^3x \left[\delta\vec{E}\cdot \underbrace{(\dot{\vec{A}} - \vec{E} + \nabla A_0)}_{\text{equation of motion}}\right] + \int dt \int_{r\to\infty} d\Omega\, r^2\,\delta E^r A_0$$

$\delta I = 0$ implies the equation of motion *provided* the boundary term is zero.

$$4\pi\delta Q \int dt A_0 : \quad \begin{cases} A_0 \sim r^n \; (n>0) & \text{badyl defined theory} \\ A_0 \sim \frac{1}{r^n} \; (n>0) & \text{boundary term is zero } \checkmark \\ & \text{“critical” behavior} \\ A_0 \sim 1 & \text{Gauge} \Rightarrow \text{Noether} \end{cases}$$

The asymptotic allowed values for A_0 are (Regge-Teitelboim classification)

$$A_0 \simeq \underbrace{\alpha_0}_{\text{Noether}} + \underbrace{\frac{\alpha_1}{r} + \frac{\alpha_2}{r^2} + \cdots}_{\text{pure gauge}}$$

In these Lectures we shall discuss gauge theories and Hamiltonian form.

- The number of degrees of freedom
- How does the Hilbert space behave under gauge transformations?
- Does Noether theorem apply to gauge symmetries?
- How do we define energy in general relativity?
- Is electric charge conservation a Noether conservation law?

Important gauge theories are

1. Yang-Mills theories (including QED)

$$I_{YM}[A^a_\mu] = -\frac{1}{4}\int \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}), \qquad \delta A^a_\mu(x) = D_\mu\lambda^a(x)$$

2. Einstein Gravity (and its generalizations, $f(R)$ gravity, Gauss-Bonnet, Chern-Simons...)

$$I_{GR}[g_{\mu\nu}] = \int \sqrt{g}(R - 2\Lambda), \qquad \delta g_{\mu\nu}(x) = \xi^\alpha g_{\mu\nu,\alpha} + \xi^\alpha{}_{,\mu} g_{\alpha\nu} + \xi^\alpha{}_{,\nu} g_{\mu\alpha}$$

3. The string worldsheet action

$$I[X^\mu, h_{\sigma\rho}] = \int \sqrt{h} h^{\sigma\rho}\partial_\sigma X^\mu \partial_\rho X^\nu \eta_{\mu\nu}, \qquad \delta X^\mu = \epsilon^\sigma \partial_\sigma X^\mu$$

Examples of Lagrangians with a gauge symmetry in particle mechanics are:

1. The parameterized non-relativistic point particle (will take us to Schroedinger equation),

$$I[q(\tau), t(\tau)] = \int \left(\frac{1}{2\dot{t}} \dot{q}^2 - \dot{t} V(q) \right) dt$$

2. The relativistic point particle (will take us to Klein-Gordon equation)

$$I[x^\mu] = -m \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau$$

One-to-one correspondence between gauge symmetries and constraints:

$$\text{QED} \quad : \quad I[A_\mu] = -\frac{1}{4}\int d^4 F^{\mu\nu}F_{\mu\nu}, \qquad (\delta A_\mu = \partial_\mu \Lambda)$$

$$= \int \left[E^i \dot{A}_i - \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) + A_0 \nabla \cdot \vec{E} \right]$$

$$\text{Yang-Mills} \quad : \quad I[A^a_\mu] = -\frac{1}{4}\int \text{Tr} F^{\mu\nu}_a F^a_{\mu\nu}, \qquad (\delta A^a_\mu = D\lambda^a)$$

$$= \int \left[E^i_a \dot{A}^a_i - \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) + A^a_0 \nabla \cdot \vec{E}_a \right]$$

$$\text{Gravity} \quad : \quad I[g] = \int \sqrt{g} R, \qquad (\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu})$$

$$= \int \left[\pi^{ij} \dot{g}_{ij} - N\mathcal{H}_\perp - N^i \mathcal{H}_i \right]$$

General structure of a gauge theory in Hamiltonian form:

$$I[p_i, q^j, \lambda^\alpha] = \int dt \left[p_i \dot{q}^i - H_0(p,q) - \lambda^\alpha \phi_\alpha(p,q) \right] \qquad \left([p_i, q^j] = \delta^j{}_i \right)$$

The λ^α appear linearly and their equations of motion are constraints

$$\phi_\alpha(p,q) = 0.$$

The theory will have a gauge symmetry if the constraints can be zero at all times,

$$\dot{\phi}_\alpha = \frac{\partial \phi_\alpha}{\partial q^i} \dot{q}^i + \frac{\partial \phi_\alpha}{\partial p_i} \dot{p}_i = [\phi_\alpha, H_0 + \lambda^\beta \phi_\beta] = 0.$$

This condition can be true, for all $\lambda^\beta(t)$, if and only if, there exists functions $f^\alpha_{\ \beta\gamma}$ and $C^\alpha_{\ \beta}$ such that,

$$\begin{aligned} [H_0, \phi_\alpha] &= C^\beta{}_\alpha \phi_\beta \\ [\phi_\alpha, \phi_\beta] &= f^\gamma{}_{\alpha\beta} \phi_\gamma \end{aligned}$$

The constraints ϕ_α are said to satisfy a "a first class algebra".

Example:

Yang-Mills theory for a Lie algebra $\mathcal{G}$ with structure constants $f^a{}_{bc}$.

$$A_\mu = A_\mu^a J_a, \qquad [J_a, J_b] = f^c{}_{ab} J_c$$

The Hamiltonian action is

$$I_{YM}[A_\mu^a] = \int \left[E_a^i \dot{A}_i^a - \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + A_0^a \, D_i E_a^i \right]$$

The constraints are:

$$\begin{aligned} \phi_a &= D_i E_a^i \\ &= \partial_i E_a^i + f^a{}_{bc} A^{a\,i} E_i^b \end{aligned}$$

and satisfy

$$[\phi_b(t,x), \phi_c(t,x')] = f^a{}_{bc} \phi_a(t,x) \delta^3(x,x').$$

General proof of gauge invariance.

Consider a Hamiltonian action of the form,

$$I[p_i, q^j, \lambda^\alpha] = \int dt \left[p_i \dot{q}^i - H_0(p,q) - \lambda^\alpha \phi_\alpha(p,q) \right].$$

If the constraints are first class, then the following transformation is a gauge symmetry of the action,

$$\begin{aligned} \delta q^i(t) &= [q^i, \phi_\alpha]\epsilon^\alpha(t) \\ \delta p_i(t) &= [p_i, \phi_\alpha]\epsilon^\alpha(t) \\ \delta \lambda^\alpha(t) &= -\dot{\epsilon}^\alpha(t) - C^\alpha_{\ \beta}\epsilon^\beta(t) - f^{\ \alpha}_{\beta\gamma}\lambda^\beta \epsilon^\gamma(t) \end{aligned}$$

where $\epsilon^\alpha(t)$ is a fully arbitrary function of time.

- In QED, the Lagrange multiplier is A_0. Recall that $\delta A_\mu = \partial_\mu \epsilon$ thus $\delta A_0 = \dot{\epsilon}$, as expected. This is an Abelian theory with $C^\alpha_{\ \beta} = 0 = f^{\ \alpha}_{\beta\gamma}$.

Proof: For any function $A(p,q)$ of the canonical variables it follows, $\delta A(q^i, p_i) = \frac{\partial A}{\partial p_i}\delta p_i + \frac{\partial A}{\partial q^i}\delta q^i = [A, \phi_\alpha]\epsilon^\alpha$. In particular,

$$\begin{aligned} \delta H_0 &= [H_0, \phi_\alpha]\epsilon^\alpha = C^\beta_{\ \alpha}\phi_\beta\,\epsilon^\alpha \\ \delta\phi_\gamma &= [\phi_\gamma, \phi_\alpha]\epsilon^\alpha = f^\beta_{\ \gamma\alpha}\phi_\beta\,\epsilon^\alpha \end{aligned}$$

The variation of the kinetic term is

$$\begin{aligned} \delta(p_i\dot{q}^i) &= \delta p_i \dot{q}^i - \dot{p}_i \delta q^i \\ &= [p_i, \phi_\alpha]\epsilon^\alpha\,\dot{q}^i - \dot{p}_i[q^i, \phi_\alpha]\epsilon^\alpha \\ &= \epsilon^\alpha\left(\frac{\partial\phi_\alpha}{\partial q^i}\dot{q}^i + \frac{\partial\phi_\alpha}{\partial p_i}\dot{p}_i\right) \\ &= \epsilon^\alpha\,\dot{\phi}_\alpha \\ &= -\dot{\epsilon}^\alpha\,\phi_\alpha \end{aligned}$$

So, the variations of H_0, ϕ_α and $p_i\dot{q}^i$ give terms proportional to the constraints.

Putting all together, the variation of the full action becomes:

$$\begin{aligned} \delta I &= \int \delta(p_i \dot{q}^i) - \delta H_0 - \lambda^\alpha \delta\phi_\alpha - \delta\lambda^\alpha \phi_\alpha \\ &= -\int \left(\dot{\epsilon}^\alpha + C^\alpha_{\ \beta}\epsilon^\beta + f^\alpha_{\ \gamma\beta}\lambda^\gamma\epsilon^\beta + \delta\lambda^\alpha \right) \phi_\alpha . \end{aligned}$$

We can choose $\delta\lambda^\alpha$ to cancel everything making the action invariant.

$$= 0$$

Observe the tight relationship between constraints, Lagrange multipliers and gauge symmetries.

$$\text{If } \lambda^\alpha \text{ was fixed} \Rightarrow \begin{cases} \text{no variation with respect to } \lambda^\alpha\text{, no constraint} \\ \delta\lambda^\alpha = 0\text{, no gauge symmetry} \end{cases}$$

Comment:

Is the scalar field action

$$I[X] = \frac{1}{2}\int \sqrt{h} h^{\mu\nu}\partial_\mu X \partial_\nu X, \quad \Rightarrow \quad \frac{1}{\sqrt{h}}\partial_\mu\left(\sqrt{h} h^{\mu\nu}\partial_\nu X\right) = 0$$

on a curved, but fixed background $h_{\mu\nu}$, gauge invariant? No. There are no constraints, no Lagrange multipliers. No gauge symmetry.

On the contrary, the same action with a dynamical metric (string worldsheet action)

$$I[X, h_{\mu\nu}] = \frac{1}{2}\int \sqrt{h} h^{\mu\nu}\partial_\mu X \partial_\nu X \Rightarrow \begin{cases} \frac{1}{\sqrt{h}}\partial_\mu\left(\sqrt{h} h^{\mu\nu}\partial_\nu X\right) = 0 \\ \partial_\mu X \partial_\nu X - \frac{1}{2} h_{\mu\nu} h^{\alpha\beta}\partial_\alpha X \partial_\beta X = 0 \\ \upuparrows \quad \text{(Virasoro constraints)} \end{cases}$$

is gauge invariant. Varying $h_{\mu\nu}$ yields constraints, and $h_{0\mu}$ are the Lagrange multipliers.

Gauge theories and time evolution. The equation of motion for a gauge theory are

$$\begin{aligned}\dot{p}_i &= [p_i, H_0] + [p_i, \phi_\alpha]\lambda^\alpha \\ \dot{q}^i &= [q^i, H_0] + [q^i, \phi_\alpha]\lambda^\alpha \\ \phi_\alpha &= 0\end{aligned}$$

We observe that the time evolution includes a gauge transformation with parameter λ^α. Given an initial condition (must satisfy $\phi_\alpha = 0$) the equations of motion have many solutions!

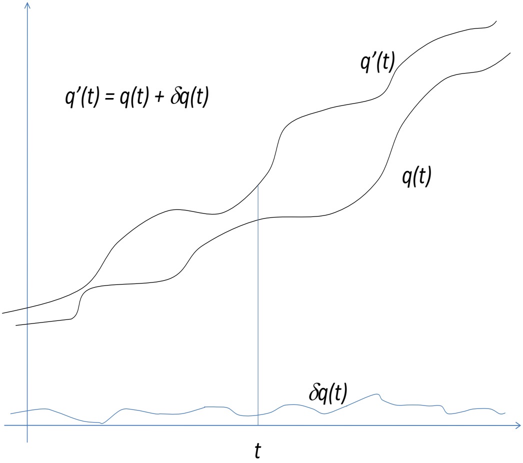
We either throw away this theory (because its time evolution is not unique) or come up with a clever interpretation:

Only combinations that do not see λ^α are physical. Different values for λ^α must be unobservable.

- The electric ($\vec{E}$) and magnetic ($\vec{B}$) fields ($F_{\mu\nu}$) in QED.
- Curvature invariants, $g^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu}R_{\mu\nu}$, ... in gravity.
- Wilson loops $Pe^{\oint_\gamma A}$ in Yang-Mills theory.

Review of first lecture:

A symmetry (Noether or Gauge, Lagrangian or Hamiltonian) is a small function δq^i that satisfies the following property. For *all* $q^i(t)$ (solutions and not solutions) we build new functions $q'^i(t) = q^i(t) + \delta q^i(t)$.



δq is a symmetry if, *for all* $q(t)$,

$$\begin{aligned} \delta I[q] &= I[q'] - I[q] \\ &= \int dt \frac{dB}{dt} \end{aligned}$$

This is a property of δq^i.

Noether Symmetries

$$\frac{dQ_a(p,q)}{dt} = [Q_a, H] = 0 \qquad \Leftrightarrow \qquad \begin{array}{l} \delta q^i = [q^i, Q_a]\rho^a \\ \delta p_i = [p_i, Q_a]\rho^a \end{array}$$

with *constant* parameters ρ^a.

Gauge Symmetries

$$\phi(q,p) = 0 \qquad \Leftrightarrow \qquad \begin{array}{l} \delta q^i = [q^i, \phi_\alpha]\epsilon^\alpha(t) \\ \delta p_i = [p_i, \phi_\alpha]\epsilon^\alpha(t) \end{array}$$

with arbitrary *time dependent* parameters $\epsilon^\alpha(t)$.

A gauge theory can also have Noether symmetries:

$$I[A] = -\frac{1}{4}\int F^{\mu\nu}F_{\mu\nu} \qquad \begin{cases} \textit{gauge}: & \delta A_\mu = \partial_\mu \epsilon(x^\nu) \\ & \\ \textit{Lorentz}: & \delta A_\mu = \rho^\nu{}_\mu A_\nu, \quad \rho^{\mu\nu} = -\rho^{\nu\mu} \\ \textit{Translations}: & \delta A_\mu = \rho^\nu \partial_\nu A_\mu \end{cases}$$

Exercise: The (classical) global group of QED (and Yang-Mills) is much larger,

$$\delta A_\mu = F_{\mu\nu}\rho^\nu(x), \qquad (\rho_{\mu,\nu} + \rho_{\nu,\mu} = \frac{1}{2}\rho^\sigma{}_\sigma \eta_{\mu\nu}) \tag{1}$$

1. Prove that (1) is a symmetry of the Maxwell action
2. Prove that (1) contains Lorentz, Translations, but also Dilatations, and Special Conformal Transformations (last two broken in QM)
3. Compute the translational Noether current ($\rho^\mu = a^\mu$)

$$J^\mu = \underbrace{\left(F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\delta^\mu_\nu\right)}_{T^\mu{}_\nu} a^\nu, \qquad T^{\mu\nu}: \begin{cases} \text{Symmetric} \\ \text{Gauge invariant} \end{cases}$$

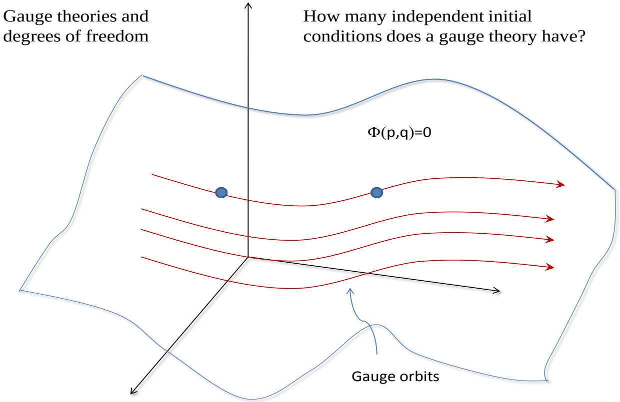
4. Why does $T^0_{\ 0}$ represent energy density? Noether point of view.

- For any conserved current J^μ ($\partial_\mu J^\mu = 0$) the following quantity is conserved,
$$Q = \int d^3x \, J^0, \qquad \frac{dQ}{dt} = 0.$$

- The conserved current due to translations a^μ is proportional to the energy-momentum tensor:
$$J^\mu_a = T^\mu_{\ \nu} a^\mu$$

- The current associated to *time translations* is then
$$J^\mu_{a^0} = T^\mu_{\ 0}.$$

- The conserved quantity, that we call energy,
$$E = \int d^3x T^0_{\ 0} = \frac{1}{2}\int d^3x(\vec{E}^2 + \vec{B}^2)$$
coincides with the Hamiltonian itself. Check the last equality for QED.



- $2N$ $\{p_i, q^i\}$'s satisfying first order equations. "$2N$ initial conditions."
- However, these initial conditions must satisfy g constraints $\phi_\alpha(p,q) = 0$. We are left with $2N - g$ independent initial conditions
- Furthermore, two initial conditions related by a gauge transformations are the same. There are g gauge transformations. Thus each gauge symmetry kills two initial conditions.

$$\begin{aligned} \text{Number of degrees of freedom} &\equiv \frac{\text{independent initial conditions}}{2} \\ &= \frac{2N - g - g}{2} \\ &= N - g. \end{aligned}$$

Examples:

- 4d Gravity: $g_{ij} = 6$ functions - 4 symmetries $= 2$
- QED : $A_i = 3$ functions - 1 symmetry $= 2$
- d-dimensional Gravity: $g_{ij} = \frac{(d-1)d}{2}$ functions - d symmetries $= \frac{d(d-3)}{2}$.
- d-dimensional Yang-Mills: $A_i^a = (d-1)N$ fields - N symmetries $= N(d-2)$

Do gauge symmetries have a Noether conserved charge?

Well, yes, but is it zero. Applying Noether theorem to the gauge symmetry yields

$$Q = \phi_\alpha = 0 \tag{2}$$

“Old” Dirac quantization condition:

Let us quantize our theory and now q^i and p_j are beautiful operators acting on a Hilbert space. What is the role of the constraints

$$\hat{\phi}_\alpha = \phi_\alpha(\hat{q}, \hat{p}) \quad ?$$

An example. In particle quantum mechanics, rotations are generated by $\vec{L} = \vec{r} \times \vec{p}$. A rotated state is $\delta|\Psi\rangle = i\vec{\alpha}\cdot\vec{L}|\Psi\rangle$

In a gauge theory, the symmetry is generated by $\hat{\phi}_\alpha$ and the “rotated” state will be

$$\delta|\Psi\rangle = \epsilon^\alpha \hat{\phi}_\alpha |\Psi\rangle$$

But we have made the assumption that gauge transformations do not affect observables. Physical states must be invariant under gauge transformations. The wave function must be invariant:

$$\phi_\alpha|\Psi\rangle = 0$$

Examples of Dirac quantization: Free relativistic particle:

$$\begin{aligned} I[X^\mu(\tau)] &= -m\int d\tau\sqrt{\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}} \\ p_\mu &= \frac{\partial L}{\partial \dot{X}^\mu} = -m\frac{\dot{X}_\mu}{\sqrt{\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu}}} \end{aligned}$$

$$\left[\begin{array}{c} \text{Invariant under} \\ \tau \to \tau' = f(\tau) \end{array}\right]$$

and it follows directly that

$$p_\mu p^\mu + m^2 = 0,$$

The Hamiltonian action is:

$$I[X,p,\lambda] = \int d\tau\left[p_\mu\dot{X}^\mu + \lambda(p^2+m^2)\right]$$

We now quantize $\hat{p}^\mu = i\frac{\partial}{\partial X^\mu}$ and Dirac condition becomes:

$$(-\Box + m^2)\psi = 0, \qquad \text{Kein-Gordon equation}$$

Non-relativistic particle (time $t(\tau)$ as a canonical variable)

$$= \int dt \left(\frac{m}{2} \left(\frac{d\vec{r}}{dt} \right)^2 - V(\vec{r}) \right) \qquad \left[\begin{array}{l} \text{Replace} \\ dt = \frac{dt}{d\tau}\, d\tau \equiv \dot{t}\, d\tau \end{array} \right]$$

$$I[\vec{r}(\tau), t(\tau)] = \int d\tau \left(\frac{m}{2} \frac{\dot{\vec{r}}^2}{\dot{t}} - \dot{t}\, V(\vec{r}) \right), \qquad \left[\begin{array}{l} \text{Invariant under} \\ \tau \to \tau' = f(\tau) \end{array} \right]$$

$$\left.\begin{array}{ll} \vec{p} & = \frac{\partial L}{\partial \dot{\vec{r}}} = m \frac{\dot{\vec{r}}}{\dot{t}}, \\ p_t & = \frac{\partial L}{\partial \dot{t}} = -\frac{m}{2} \frac{\dot{\vec{r}}^2}{\dot{t}^2} - V(\vec{r}). \end{array}\right\} \quad p_t + \frac{1}{2m} \vec{p}^2 + V(\vec{r}) = 0$$

We quantize

$$p_t = -i \frac{\partial}{\partial t}, \qquad \vec{p} = i\nabla$$

and the Dirac condition becomes:

$$i \frac{\partial \Psi}{\partial t} = \left(-\frac{1}{2m} \nabla^2 + V \right) \Psi, \qquad \text{Schroedinger equation}$$

The string worldsheet action. This is a field equation with infinite number of degree of freedom. It requires a detailed analysis:

$$I[X, h_{\mu\nu}] = \frac{1}{2}\int \sqrt{h} h^{\mu\nu}\partial_\mu X \partial_\nu X \;\Rightarrow\; \begin{cases} \frac{1}{\sqrt{h}}\partial_\mu\left(\sqrt{h}h^{\mu\nu}\partial_\nu X\right) = 0 \\ \partial_\mu X\partial_\nu X - \frac{1}{2}h_{\mu\nu}h^{\alpha\beta}\partial_\alpha X\partial_\beta X = 0 \\ \quad\upuparrows \;\; \text{(Virasoro constraints)} \end{cases}$$

$$: \partial_\mu X\partial_\nu X - \frac{1}{2}h_{\mu\nu}h^{\alpha\beta}\partial_\alpha X\partial_\beta X : \;\Rightarrow\; \hat{L}_n,\; \hat{\bar{L}}_n$$

Dirac "improved condition" becomes

$$L_n|\Psi\rangle = 0, \qquad n > 0$$

Yet another class of symmetry: Trivial symmetries. Let us denote all canonical coordinates $q^i, p_j \equiv z^a$ where $a = 1, 2, ..., 2N$. The action is $I[z^a]$.

Let $w^{ab}(z) = -w^{ba}(z)$ an arbitrary $z-$dependent antisymmetric tensor. The following transformation is a symmetry of the action:

$$\delta z^a = w^{ab}\frac{\delta I[z]}{\delta z^b}, \qquad \Rightarrow \qquad \delta I = \frac{\delta I[z]}{\delta z^a} w^{ab}\frac{\delta I[z]}{\delta z^b} = 0.$$

Of course these transformations –that exists for any action– cannot be interesting! They have neither conserved charges nor constraints. The true set of symmetries of an action is "the quotient space of all symmetries divided by the set of trivial symmetries".

Note that these transformations are proportional to the equations of motion,

$$\delta z^a = 0 \qquad (on - shell)$$

The converse is also true. If $\delta z^a \approx 0$, then it is trivial.

Examples of trivial symmetries.

Three dimensional Gravity in vielbein formalism:

$$I[w,e] = \int \epsilon_{abc} R^{ab} \wedge e^c \qquad \left(= \int \sqrt{g} R\right)$$

The equations of motion are

$$R^{ab} = 0, \qquad D \wedge e^a = 0$$

This action is invariant under the following gauge symmetries

1. Lorentz Rotations: $\delta_w e^a_\mu = \epsilon^a{}_b e^b, \qquad \delta w^{ab} = -D\epsilon^{ab}$
2. Translations: $\delta_\lambda e^a_\mu = D_\mu \lambda^a, \qquad \delta w^{ab} = 0 \qquad (D \wedge R^{ab} = 0)$
3. Diffeomorphisms: $\delta_\xi e^a = \xi^\alpha e^a_{\mu,\alpha} + \xi^\alpha{}_{,\mu} e^a{}_\alpha$.

A new symmetry for gravity...? No. 3 can be expressed in terms of 1 and 2, plus a trivial term

$$\xi^\alpha e^a_{\mu,\alpha} + \xi^\alpha{}_{,\mu} e^a{}_\alpha = D_\mu(e^a_\nu \xi^\nu) + \xi^\nu w^a{}_{b\nu} e^b{}_\mu \quad + \quad (D \wedge e^a)_{[\mu\nu]} \xi^\nu$$

Topological Chern-Simons theories, gauge transformations, and diffeomorphisms. Let a gauge field

$$A = A^a_\mu dx^\mu$$

and build the topological functionals (metric independent):

$$\begin{aligned} I_3[A] &= \int \mathrm{Tr}(AdA + \frac{2}{3}AAA) \\ I_5[A] &= \int \mathrm{Tr}(AdAdA + \frac{2}{3}AAAdA + \frac{1}{5}AAAAA) \\ I_7[A] &= \ldots \end{aligned}$$

These theories are gauge *and* diff invariant:

$$\delta A^a_\mu = D_\mu \lambda^a, \qquad \delta A^a_\mu = \xi^\nu \partial_\nu A_\mu + \xi^\nu{}_{,\mu} A^a_\nu$$

- In three dimensions, diffs are contained in the gauge group.
- In higher dimensions ($F \neq 0$) "spatial" diffs ξ^i are independent, while "temporal" ξ^0 diffs are contained in the gauge group.

More on topological diff invariance. An application to Horava gravity.

Consider a theory of the form:

$$I[p,q,\lambda] = \int p_i \dot{q}^i - \lambda^\alpha \phi_\alpha(p,q)$$

If the constraints are first class $[\phi_\alpha, \phi_\beta] = f^{\gamma}_{\ \alpha\beta}\phi_\gamma$, then we know it has a gauge symmetry.

This action is also invariant under time reparameterizations (because $H_0 = 0$)

$$\begin{aligned} \delta q^i &= \eta(t)\dot{q}^i \\ \delta p_i &= \eta(t)\dot{p}_i \\ \delta\lambda^\alpha &= (\eta\lambda^\alpha)^{\cdot} \end{aligned}$$

Proof:

$$\begin{aligned} \delta q^i &= \eta(t)\dot{q}^i \\ \delta p_i &= \eta(t)\dot{p}_i \\ \delta\lambda^\alpha &= (\eta\lambda^\alpha)^\cdot \end{aligned}$$

We first compute the variation of the constraint:

$$\begin{aligned} \delta\phi_\alpha &= \frac{\partial\phi_\alpha}{\partial q^i}\dot{q}^i\eta(t) + \frac{\partial\phi_\alpha}{\partial p_i}\dot{p}_i\eta(t) \\ &= \dot{\phi}_\alpha\eta(t) \end{aligned}$$

and the variation of the full action is:

$$\begin{aligned} \delta I &= \int \delta p_i\dot{q}^i - \dot{p}_i\delta q^i - \delta\lambda^\alpha\phi_\alpha - \lambda^\alpha\delta\phi_\alpha \\ &= \int \eta\dot{p}_i\dot{q}^i - \dot{p}_i\eta\dot{q}^i - (\eta\lambda^\alpha)^\cdot\phi_\alpha - \lambda^\alpha\eta\dot{\phi}_\alpha \\ &= \int -\frac{d}{dt}(\eta\lambda^\alpha\phi_\alpha) \qquad \checkmark \end{aligned}$$

Is this symmetry new? Not really.

$$\begin{aligned} \delta q^i &= \eta(t)\dot{q}^i \\ \delta p_i &= \eta(t)\dot{p}_i \\ \delta\lambda^\alpha &= (\eta\lambda^\alpha)^\cdot \end{aligned} \qquad \begin{aligned} \delta q^i &= [q^i, \phi_\alpha]\epsilon^\alpha \\ \delta p_i &= [p_i, \phi_\alpha]\epsilon^\alpha \\ \delta\lambda^\alpha &= \dot{\epsilon}^\alpha + f^\alpha_{\beta\gamma}\lambda^\beta\epsilon^\gamma \end{aligned}$$

- The left transformations are contained in the right transformations when $\epsilon^\alpha = \eta\lambda^\alpha$, plus the use of the equations of motion. They are related by a trivial symmetry.
- What happens if the constraints are second class?

$$[\phi_\alpha, \phi_\beta] = C_{\alpha\beta}, \qquad \det(C_{\alpha\beta}) \neq 0$$

  The right symmetry ceases to exist. But the left symmetry still exists.
- However, this symmetry is now trivial: $0 = \dot{\phi}_\alpha = [\phi_\alpha, \phi_\beta]\lambda^\beta = C_{\alpha\beta}\lambda^\beta$ implies $\lambda^\alpha = 0$. It follows $\dot{q}^i = \dot{p}_i = 0$ and the transformations are zero on-shell, hence trivial.
- Horava has modified the Hamiltonian constraint to make GR renormalizable. However, $\mathcal{H}_{Horava}$ is second class. Hence diffeomorphisms are trivial, not relevant.

General Relativity in Hamiltonian form: Let M a manifold with local coordinates $x^\mu = (t, x^i)$. t is timelike, x^i are spacelike. Any metric $g_{\mu\nu}(x)$ can be decomposed in the form

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

where $N(t,x), N^i(t,x)$ are functions replacing $g_{0\mu}$. The Einstein-Hilbert Lagrangian Lagrangian action can be written as

$$\int d^4x \sqrt{{}^{(4)}g}\, {}^{(4)}R \;=\; \int dt\, N \int d^3x \sqrt{|g_{ij}|} \left[\underbrace{K^{ij}K_{ij} - K^2}_{Kinetic\;\; term} + \underbrace{R(g_{ij})}_{Interaction} \right] + \mathrm{bt}$$

$$K_{ij} \;\equiv\; \frac{1}{2N}(-\dot{g}_{ij} + N_{i/j} + N_{j/i})$$

- This Lagrangian (called ADM) is quadratic in $\dot{g}_{ij}$.
- Does not contain time derivatives of N, N^i (analogous to A_0 in QED).
- We shall do a Lagrange transformation with respect to $\dot{g}_{ij}$.

Define momenta for g_{ij} in the usual way:

$$\pi^{ij} = \frac{\partial L}{\partial \dot{g}_{ij}} = -\sqrt{g}(K_{ij} - g_{ij}K)$$

This equation can be inverted to express $\dot{g}_{ij}$ in terms of π^{ij}. The Hamiltonian becomes a linear combination

$$\begin{aligned} H &= \pi^{ij}\dot{g}_{ij} - L \\ &= N\mathcal{H} + N^i\mathcal{H}_i, \end{aligned}$$

where

$$\mathcal{H}(\pi, g) = \frac{1}{\sqrt{g}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2) - \sqrt{g}R \quad (3)$$

$$\mathcal{H}_i(\pi, g) = -2\pi^j{}_{i/j} \quad (4)$$

are functions of π^{ij}, g_{ij}.

The Hamiltonian action of GR has the structure of a gauge theory:

$$I[\pi^{ij}, g_{ij}, N, N^i] = \int dt d^3x \left[\pi^{ij}\dot{g}_{ij} - \underbrace{(N\mathcal{H} + N^i\mathcal{H}_i)}_{Hamiltonian} \right]$$

Variation with respect to N, N^i give rise to the constraints $\mathcal{H} = 0, \quad \mathcal{H}_i = 0$;
They form a closed algebra (hence a gauge symmetry exists)

$$\begin{aligned}
[\mathcal{H}(x), \mathcal{H}(y)] &= g^{ij}(\mathcal{H}_i(x) + \mathcal{H}_i(y))\partial_j\delta^3(x,y) \\
[\mathcal{H}(x), \mathcal{H}_i(y)] &= \partial_i(\mathcal{H}(x)\delta^3(x,y)) \\
[\mathcal{H}_i(x), \mathcal{H}_j(y)] &= (\mathcal{H}_j(x) + \mathcal{H}_i(y))\partial_j\delta^3(x,y)
\end{aligned}$$

the dynamical equations of motion are:

$$\begin{aligned}
\dot{g}_{ij}(x) &= \frac{\delta H_0}{\delta\pi^{ij}(x)}, \\
\dot{\pi}^{ij}(x) &= -\frac{\delta H_0}{\delta g_{ij}(x)}
\end{aligned} \quad , \qquad H_0 = \int d^3x[N\mathcal{H} + N^i\mathcal{H}_i]$$

Hence, as in any field theory we need to compute functional derivatives.

A quick reminder of a functional derivative:

$$F[\Phi] = \frac{1}{2}\int d^3x \nabla\Phi(x)\cdot\nabla\Phi(x)$$

then

$$\begin{aligned}\delta F[\Phi] &= \int d^3x \nabla\Phi\cdot\nabla\delta\Phi \\ &= -\int d^3x \nabla^2\Phi\,\delta\Phi + \underbrace{\int d\vec{S}\cdot\nabla\Phi\,\delta\Phi}_{\text{boundary term}}\end{aligned}$$

If the boundary term is zero, then the functional derivative is well-defined and equal to:

$$\frac{\delta F[\Phi]}{\delta\Phi(x)} = -\nabla^2\Phi.$$

The lesson here is that functional derivatives are not automatically well-defined. The boundary term must vanish.

The variation of the GR Hamiltonian functional has the following expression:

$$\delta H_0 = \underbrace{\int d^3x \left(A_{ij}\delta\pi^{ij} + B^{ij}\delta g_{ij}\right)}_{\text{volume pieces} \quad \checkmark} +$$

$$+ \underbrace{\int dS_l \left[G^{ijkl}(N\delta g_{ij;k} - N_{,k}\delta g_{ij}) + 2N_i\delta\pi^{il} + (2N^i\pi^{kl} - N^i\pi^{ik})\delta g_{jk}\right]}_{\text{boundary term} \equiv \delta B}$$

- Recall that N, N^i are arbitrary. So generically $\delta B \neq 0$
- The action does not have an extremum...
- The idea is to "pass B to the other side" and build a new Hamiltonian with well-defined variations,

$$H \equiv H_0 - B, \qquad \delta H = \int d^3x \left(A_{ij}\delta\pi^{ij} + B^{ij}\delta g_{ij}\right)$$

- This trick preserves the equations of motion $\dot{g}_{ij} = A_{ij}, \ \dot{\pi}^{ij} = -B^{ij}$ but,
- It changes the interpretation of those diffeomorphisms with $B \neq 0$.

According to our discussion so far:

"Gauge transformations do not change the physical state"
They are generated by constraints $\phi_\alpha(p, q) = 0$, and
leave physical states invariant: $\phi_\alpha(\hat{p}, \hat{q})|\Phi\rangle = 0$.

However, some diffeomorphisms with particular asymptotic values of N, N^i are not generated by constraints:

$$\begin{aligned} H[N] &= \int d^3x (N\mathcal{H} + N^i\mathcal{H}_i) - B[N] \\ &= \quad 0 \quad - \quad B[N] \end{aligned}$$

We conclude:

All gauge transformations are equal
but some are more equal than others

"Gauge Farm"

Coordinate transformations in General Relativity are generated by

$$H[N] = \underbrace{\int d^3x(N\mathcal{H} + N^i\mathcal{H}_i)}_{\approx 0} - B[N]$$

There are three classes of coordinate transformations

1. $B[N] = 0$. These are true "gauge transformations" generated by a constraint $H = 0$. They do not change the physical state

2. $B[N] \neq 0$. These are not "gauge transformations". They are generated by a non-zero quantity and are a global symmetry with a conserved Noether charge B. They do change the physical state: $H|\Psi\rangle \neq 0$.

3. If $B[N] = \infty$ these are not symmetries of the system (although see calculation of anomalies in AdS/CFT).

Now, could you please stop talking and calculate this boundary term?

We shall do it explicitly for AdS_3 described by the action

$$I[g_{\mu\nu}, \text{matter}] = \frac{1}{16\pi G}\int d^3x\sqrt{g}(R - 2\Lambda) + \text{matter}.$$

We consider this case because it is nice, simple and has a lot of structure (Virasoro algebra, a central charge). Most importantly, the speaker had all the equations at hand. The same can be done with minor modifications for asymptotically flat spacetimes, and in any number of dimensions.

We need to compute integrals at infinity which look like,

$$\int_{r\to\infty} d\Omega r^{d-2} n^i N^j \delta g_{ij} \qquad \text{(plus many others)}$$

To determine whether this is zero or not, we need to know how fast/slow the fields $N, N^i, \delta g_{ij}$ decay as $r \to \infty$.

The Regge-Teitelboim algorithm is as follows: The "vacuum" solution to three-dimensional gravity is AdS_3 ($\Lambda = -1$)

$$ds^2_{(0)} = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\phi^2$$

This is the relevant solution when there is no matter, no sources. This vacuum state has 6 isometries

$$\begin{aligned}
\xi_1^\mu &= [1,0,0], \qquad \text{time translations} \\
\xi_2^\mu &= [0,0,1], \qquad \text{rotations} \\
\xi_3^\mu &= \left[\frac{-r\sin t\cos\phi}{\sqrt{1+r^2}}, \sqrt{1+r^2}\cos t\cos\phi, \frac{\sqrt{1+r^2}}{r}\sin t\sin\phi\right] \\
\xi_4^\mu &= \left[\frac{-r\sin t\sin\phi}{\sqrt{1+r^2}}, \sqrt{1+r^2}\cos t\sin\phi, \frac{\sqrt{1+r^2}}{r}\sin t\sin\cos\right] \\
\xi_5^\mu &= \ldots \\
\xi_6^\mu &= \ldots
\end{aligned}$$

We call this set of vectors ξ_a^μ, a=1,2..6. They all satisfy the Killing equation

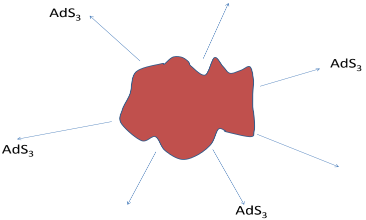
$$\mathcal{L}_{\xi_a} g^{(0)}_{\mu\nu} = 0.$$

- They satisfy a Lie algebra:

$$[\xi_a, \xi_b]^\mu \;\; = \;\; f^c{}_{ab} \xi_c^\mu$$

- This algebra can be decompose into a direct product $SL(2, \Re) \times SL(2, \Re)$.

With matter the geometry deviates from AdS_3 in the bulk, but far away we expect to get back to AdS_3. The question is how fast. For large r, the source looks like a point with metric,



$$ds^2 = -(-M + r^2)dt^2 + \frac{dr^2}{-M + r^2 + \frac{J^2}{4r^2}} + Jdtd\varphi + r^2 d\varphi^2$$

This gives us some reasonable fall off behavior for the metric functions.

But we would like the asymptotics to be AdS_3 invariant. if we act with the 6 AdS_3 Killing vectors ξ^μ_a on this metric, we generate the following asymptotic solution:

$$\begin{aligned}
g_{tt} &\sim r^2 + f_{tt}(t,\phi) + \cdots\,, \\
g_{tr} &\sim 0 + f_{tr}(t,\phi)/r^3 + \cdots\,, \\
g_{t\phi} &\sim 0 + f_{t\phi}(t,\phi)/r^3 + \cdots\,, \\
g_{rr} &\sim 1/r^2 + f_{rr}(t,\phi)/r^4 + \cdots\,, \\
g_{r\phi} &\sim 0 + f_{r\phi}(t,\phi)/r^3 + \cdots\,, \\
g_{\phi\phi} &\sim r^2 + f_{\phi\phi}(t,\phi) + \cdots\,,
\end{aligned}$$

Thus, at infinity, the metric behaves as

$$
\begin{aligned}
g_{tt} &\sim r^2 + f_{tt}(t,\phi) + \cdots \,, \\
g_{tr} &\sim 0 + f_{tr}(t,\phi)/r^3 + \cdots \,, \\
g_{t\phi} &\sim 0 + f_{t\phi}(t,\phi)/r^3 + \cdots \,, \\
g_{rr} &\sim 1/r^2 + f_{rr}(t,\phi)/r^4 + \cdots \,, \\
g_{r\phi} &\sim 0 + f_{r\phi}(t,\phi)/r^3 + \cdots \,, \\
g_{\phi\phi} &\sim r^2 + f_{\phi\phi}(t,\phi) + \cdots \,,
\end{aligned}
$$

and the fluctuations,

$$
\begin{aligned}
\delta g_{tt} &\sim f_{tt}(t,\phi) + \cdots \,, \\
\delta g_{tr} &\sim f_{tr}(t,\phi)/r^3 + \cdots \,, \\
\delta g_{t\phi} &\sim f_{t\phi}(t,\phi)/r^3 + \cdots \,, \\
\delta g_{rr} &\sim f_{rr}(t,\phi)/r^4 + \cdots \,, \\
\delta g_{r\phi} &\sim f_{r\phi}(t,\phi)/r^3 + \cdots \,, \\
\delta g_{\phi\phi} &\sim f_{\phi\phi}(t,\phi) + \cdots \,,
\end{aligned}
$$

The functions $f_{\mu\nu}(t,\varphi)$ are arbitrary. These conditions define the phase space.

We thus have all the ingredients to compute the boundary term,

$$\delta B[N, N^i] = \int dS_l \Big[G^{ijkl}(N\delta g_{ij;k} - N_{,k}\delta g_{ij}) + 2N_i\delta\pi^{il} + (2N^i\pi^{kl} - N^i\pi^{ik})\delta g_{jk}$$

The interpretation:
The full generator of 'gauge transformations' (diffeomorphisms) is then

$$H[N, N^i] = \int d^3x(N\mathcal{H} + N^i\mathcal{H}_i) - B[N, N^i],$$

This object generates a diffeomorphisms with parameters N, N^i ($\Leftrightarrow \xi^\mu$ entering in $x'^\mu = x^\mu + \xi^\mu$).

Black hole charges: Consider the black hole metric, which has specific functions $f_{\mu\nu}$.

Diffeomorphisms approaching a *constant* time translation at infinity:

$$N \to 1, \qquad B[1,0] = -M \qquad H = M \qquad \text{Energy}$$

Diffeomorphisms approaching a *constant* rotation translation at infinity:

$$N^\phi \to 1, \qquad B[0,1] = -J \quad \text{Angular Momentum}$$

Symmetries, once again... One the strongest pieces of evidence for this conjecture comes from the symmetries of both theories:

Isometries of AdS_d. In d dimensions find the vectors ξ leaving the AdS metric invariant:

$$\mathcal{L}_\xi g^{AdS}_{\mu\nu} = 0$$

(anti-de Sitter group).

Go down one dimension and find the conformal isometries of flat space .

$$\mathcal{L}_\chi \eta_{ij} = \Omega \eta_{ij}$$

the vectors χ define the conformal group.

It turns out that these algebras are the same. There is a one-to-one correspondence between the vectors ξ and the vectors χ

But in AdS_3/CFT_2 there seems to be a counterexample:

- The AdS_3 Killing equation has six vectors, spanning the Lie algebra $SL(2,\Re)^2$
- On the other hand, the conformal Killing equation in 2-dimensions has an infinite number of solutions! Take

$$ds^2 = dzd\bar{z}$$

The transformation $z = f(z')$ and $\bar{z} = \bar{f}(\bar{z}')$ yields

$$ds^2 = |\partial f|^2 dz' d\bar{z}'$$

which is the same metric, up to a conformal factor.
- The functions f and $\bar{f}$ are arbitrary

$$f(z) = a_0 z + a_1 z^2 + a_3 z^3 + \cdots$$

each term is an independent transformation.

Where are the missing 'isometries' in AdS_3?

In 1986, Brown and Henneaux discover that AdS_3 has a conformal symmetry with a central charge. Now (Maldacena 1997) we understand this symmetry as the first example of a deep relationship between anti-de Sitter spaces and conformal field theories. Consider the conformal algebra

$$\begin{aligned}[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ [\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\end{aligned}$$

The modes $L_{0,\pm1}, \bar{L}_{0,\pm1}$ form a closed subalgebra,

$$\begin{aligned}[L_1, L_{-1}] &= 2L_0 & \qquad [\bar{L}_1, \bar{L}_{-1}] &= 2\bar{L}_0 \\ [L_{\pm1}, L_0] &= \pm L_{\pm1} & \qquad [\bar{L}_{\pm1}, \bar{L}_0] &= \pm \bar{L}_{\pm1}\end{aligned}$$

This subalgebra, called $SL(2,\Re) \times SL(2,\Re)$, is exactly equivalent to the 6 Killing vectors ξ_a^μ of AdS_3.

Could it be that our boundary conditions are invariant under a larger group of transformations? Will these extra transformations have finite charges?

We have considered the class of metrics such that for large r

$$
\begin{aligned}
g_{tt} &\sim r^2 + f_{tt}(t,\phi) + \cdots , \\
g_{tr} &\sim 0 + f_{tr}(t,\phi)/r^3 + \cdots , \\
g_{t\phi} &\sim 0 + f_{t\phi}(t,\phi)/r^3 + \cdots , \\
g_{rr} &\sim 1/r^2 + f_{rr}(t,\phi)/r^4 + \cdots , \\
g_{r\phi} &\sim 0 + f_{r\phi}(t,\phi)/r^3 + \cdots , \\
g_{\phi\phi} &\sim r^2 + f_{\phi\phi}(t,\phi) + \cdots ,
\end{aligned}
$$

These class of metrics are AdS_3 invariant in the sense that any diffeomorphism $\xi^\mu \to \xi_a^\mu$ leave them invariant.

We know ask the opposite question: Are there other change of coordinates such that these conditions are invariant? Consider a vector

$$
\xi^\mu = \left(a_t(t,\varphi) + \frac{b_t(t,\varphi)}{r^2} \; , \; a_r(t,\varphi) r \; , \; a_\varphi(t,\varphi) + \frac{b_\varphi(t,\varphi)}{r^2} \right)
$$

This vector is much more general than the Killing vectors ξ_a^μ. Act with these vectors on the boundary conditions to obtain:

$$
Lg_{a\ \ b} =
\left[\left[\left(-2\,\mathrm{ar}(t,\phi) - 2\left(\frac{\partial}{\partial t}\mathrm{at}(t,\phi)\right)\right) r^2 - 2\left(\frac{\partial}{\partial t}\mathrm{bt}(t,\phi)\right), \frac{\left(\frac{\partial}{\partial t}\mathrm{ar}(t,\phi)\right) + 2\,\mathrm{bt}(t,\phi)}{r},\right.\right.
$$

$$
\left.\left(\left(\frac{\partial}{\partial t}\mathrm{aphi}(t,\phi)\right) - \left(\frac{\partial}{\partial \phi}\mathrm{at}(t,\phi)\right)\right) r^2 + \left(\frac{\partial}{\partial t}\mathrm{bphi}(t,\phi)\right) - \left(\frac{\partial}{\partial \phi}\mathrm{bt}(t,\phi)\right)\right]
$$

$$
\left[\frac{\left(\frac{\partial}{\partial t}\mathrm{ar}(t,\phi)\right) + 2\,\mathrm{bt}(t,\phi)}{r}, 0, \frac{-2\,\mathrm{bphi}(t,\phi) + \left(\frac{\partial}{\partial \phi}\mathrm{ar}(t,\phi)\right)}{r}\right]
$$

$$
\left[\left(\left(\frac{\partial}{\partial t}\mathrm{aphi}(t,\phi)\right) - \left(\frac{\partial}{\partial \phi}\mathrm{at}(t,\phi)\right)\right) r^2 + \left(\frac{\partial}{\partial t}\mathrm{bphi}(t,\phi)\right) - \left(\frac{\partial}{\partial \phi}\mathrm{bt}(t,\phi)\right),\right.
$$

$$
\left.\left.\frac{-2\,\mathrm{bphi}(t,\phi) + \left(\frac{\partial}{\partial \phi}\mathrm{ar}(t,\phi)\right)}{r}, \left(2\,\mathrm{ar}(t,\phi) + 2\left(\frac{\partial}{\partial \phi}\mathrm{aphi}(t,\phi)\right)\right) r^2 + 2\left(\frac{\partial}{\partial \phi}\mathrm{bphi}(t,\phi)\right)\right]\right]
$$

We conclude that a much larger group of transformations leave the boundary conditions invariant.

We shall now pass to a different coordinate system where the Brown-Henneaux conditions can be solved in a simple way, and the conformal symmetry exhibited in an explicit way.

Chiral coordinates and Brown-Henneaux conformal symmetry.

$$\begin{aligned} ds^2 &= -\left(1+r^2\right)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\varphi^2 \\ &\to r^2(-dt^2+d\varphi^2) + \frac{dr^2}{r^2} \\ &= e^{2\rho}dzd\bar{z} + d\rho^2, \qquad [z = t+\varphi, \quad \bar{z} = -t+\varphi, \quad e^{\rho} = r]. \end{aligned}$$

In these coordinates, the Brown-Henneaux boundary conditions read:

$$ds^2 = e^{2\rho}dzd\bar{z} + d\rho^2 + \underbrace{T(z)dz^2 + \bar{T}(\bar{z})d\bar{z}^2} + \cdots$$

where $T(z)$ and $\bar{T}(\bar{z})$ are arbitrary functions of their arguments. Furthermore, the larger group that leaves these metrics invariant are (pseudo) conformal transformations

$$z \to f(z), \qquad \bar{z} \to g(\bar{z}).$$

acting non-trivially! In AdS/CFT language, $T(z)$, $\bar{T}(\bar{z})$ are vevs.

A coordinate transformation with lots of interpretation. Chiral case ($\bar{T} = 0$).

$$ds^2 \;=\; e^{2\rho} dz d\bar{z} + d\rho^2 + T(z) dz^2/\alpha$$

(incidentally, this ↑ is an exact solution. α =constant)

$$z = f(z') \quad \rightarrow \qquad = \; e^{2\rho}\partial' f \, dz' d\bar{z} + d\rho^2 + T(z)(\partial' f)^2 dz'^2/\alpha$$

We can eliminate the Jacobian in the first term by doing

$$e^{2\rho}\partial' f = e^{2\rho'}, \qquad \bar{z} = \bar{z}' - \frac{1}{2} e^{-2\rho'} \frac{\partial'^2 f}{\partial' f}.$$

and go back to exactly metric we started from

$$ds^2 = e^{2\rho'} dz' d\bar{z}' + d\rho'^2 + T'(z') dz'^2/\alpha,$$

where –note the Schwarz derivative(!)–

$$T'(z') = T(z) \left(\partial' f\right)^2 - \frac{\alpha}{2}\{f, z'\}.$$

This would be meaningless, if it wasn't for the fact $z = f(z')$ is not trivial! The generator is non-zero, and equal to $T(z)$.

Finally, we compute the boundary term associated to conformal transformations:

$$B[f(z)] = T(z)$$

The algebra of diffeomorphisms allow the calculation of $[T, T]$ and fixes the value of α such that T is energy-momentum. With the expansion in modes,

$$T(z) = \sum \frac{L_n}{z^{n+2}}$$

one finds the Virasoro algebra,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$

with

$$c = \frac{3\ell}{2G}.$$

G and ℓ are parameters appearing in the 3d action,

$$I = \frac{1}{16\pi G} \int \sqrt{g} \left(R - \frac{2}{\ell^2} \right)$$

- The asymptotic symmetry group of three dimensional gravity is not $SL(2,\Re)^2$ but the conformal group with a non-zero central charge.

- Now, AdS/CFT [Maldacena (1997)] suggests a stronger statement,

Gravity on AdS_3 is dual to a CFT_2

- Caution: AdS/CFT applies to the whole of string theory. But perhaps AdS_3/CFT_2 represents an isolated self-consistent island? (Note Chern-Simons formulation and WZW models)

- Note: Brown-Henneaux does not give information on the CFT theory, only the symmetry. It is like knowing $[L_i, L_j] = \epsilon_{ijk} L_j$ but not $\vec{L} = \vec{r} \times \vec{p}$ (or any other).

We would like to compute the black hole entropy by counting states in the CFT.

- The back hole is asymptotically anti-de Sitter with

$$\ell MG = L_0 + \bar{L}_0, \qquad JG = L_0 - \bar{L}_0.$$

- We expect the CFT_2 central charge to be the Brown-Henneaux one (up to sub-leading quantum corrections).
- Now the magic comes from the power of Cardy formula. For **any** unitary modular invariant 2d CFT, the number of states consistent with given (large) $L_0, \bar{L}_0$ is

$$\rho(T, \bar{T}) = e^{2\pi\sqrt{\frac{c}{6}L_0}+2\pi\sqrt{\frac{c}{6}\bar{L}_0}}.$$

- Plug the black hole values for $L_0, \bar{L}_0$ and obtain exactly Bekenstein-Hawking entropy [Strominger (1997)]

$$\rho(M, J) = e^{\frac{2\pi r_+}{4G}}.$$

- Note the crucial role of the central charge c.

Thank you