(Constrained) Quantization Without Tears

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ABSTRACT

An alternative to Dirac's constrained quantization procedure is explained.

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ABSTRACT

An alternative to Dirac's constrained quantization procedure is explained.

To accomplish conventional and elementary quantization of a dynamical system, one is instructed to: begin with a Lagrangian, eliminate velocities in favor of momenta by a Legendre transform that determines the Hamiltonian, postulate canonical brackets among coordinates and momenta and finally define dynamics by commutation with the Hamiltonian. But this procedure may fail for several reasons: it may not be possible to solve for the velocities in terms of the momenta, or it may be that the Hamiltonian equations do not reproduce the desired dynamical equations. In such cases one is dealing with so-called "singular" Lagrangians and "constrained" dynamics. Almost half a century ago Dirac developed his method for handling this situation[1] and since that time the subject has defined an area of specialization in mathematical physics, as is put into evidence by a recent monograph[2] and by this series of workshops[3].

While Dirac's approach and subsequent developments can cope with most models of interest, my colleague Ludwig Faddeev and I realized that in many instances Dirac's method is unnecessarily cumbersome and can be streamlined and simplified. We have advertised[4] an alternative approach, based on Darboux's theorem, wherein one arrives at the desired results — formulas for brackets and for the Hamiltonian — without following Dirac step by step.

Very specifically, two aspects of the Dirac procedure are avoided. First, when it happens that the Lagrangian L depends linearly on the velocity $\dot{\xi}^i$ for one of the dynamical variables ξ^i, or even is independent of $\dot{\xi}^i$, the attempt to define the canonical momentum $\Pi_i = \frac{\partial L}{\partial \dot{\xi}^i}$, and to eliminate $\dot{\xi}_i$ in favor of Π_i obviously fails. In the Dirac procedure, one nevertheless defines a canonical momentum and views the $\dot{\xi}^i$-independent expression $\frac{\partial L}{\partial \dot{\xi}^i}$ as a constraint on Π_i. In our method, such constraints are never introduced. Second, in the Dirac procedure constraints are classified and distinguished as first class or second class, primary or secondary. This distinction is not made in our method; all constraints are held to the same standard.

It is therefore clear that our approach eliminates useless paperwork, and here I shall give a description with the hope that this audience of specialists will appreciate the economy of our proposal and will further adopt and disseminate it.

I shall use notation appropriate to a mechanical system, with coordinates labeled by $\{i, j, \ldots\}$ taking values in a set of integers up to N, and a summation convention for repeated indices. Field theoretic generalization is obvious: the discrete quantities $\{i, j, \ldots\}$ become continuous spatial variables. Time dependence of dynamical variables is not explicitly indicated since all quantities are defined at the same time, but time-differentiation is denoted by an over-dot. Although the language of quantum mechanics is used, with $\hbar$ scaled to unity, ("commutation," *etc.*) ordering issues are not addressed — so more properly speaking we are describing a classical Hamiltonian reduction of dynamics. Grassmann variables are not considered, since that complication is a straightforward generalization. Finally total time derivative contributions to Lagrangians are omitted whenever convenient.

Our starting point is a first-order Lagrangian formulation for the dynamics of interest; *i.e.* we assume that the Lagrangian is at most linear in time derivatives. This is to be contrasted with the usual approach, where the starting point is a second-order Lagrangian, quadratic in time-derivatives, and a first-order Lagrangian is viewed as "singular" or "constrained." In fact, just because dynamics is described by first-order differential equations, it does not mean that there are constraints, and this is a point we insist upon and we view the conventional position to be inappropriate.

Indeed there are many familiar and elementary dynamical systems that are first-order, without there being any constraints: Lagrangians for the Schrödinger equation and the Dirac equation are first-order in time derivatives; in light-cone quantization, where $x^+ \equiv \frac{1}{\sqrt{2}}(t + x)$ is the evolution coordinate, dynamics is first-order in this "time;" the most compact description of chiral bosons in two space-time dimensions is first order in time[5]. It is clear that characterizing any of these systems as "singular" or "constrained" reflects awkward mathematics rather than physical fact.

Moreover, a conventional second order Lagrangian can be converted to first-order form by precisely the same Legendre transform used to pass from a Lagrangian to a Hamiltonian. The point is that the formula

$$H = \frac{\partial L}{\partial \dot{q}}\,\dot{q} - L \tag{1}$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} \tag{2}$$

may also be read in the opposite direction,

$$L(p, q) = p\,\dot{q} - H(p, q) \tag{3}$$

and it is straightforward to verify that Euler-Lagrange equations for the first-order Lagrangian $L(p, q)$ coincide with the Hamiltonian equations based on H(p,q). Thus given a conventional Hamiltonian description of dynamics, we can always construct a first-order Lagrangian whose *configuration space* coincides with the Hamiltonian *phase space*.

We begin therefore with a general first-order Lagrangian.

$$L = a_i(\xi)\dot{\xi}^i - V(\xi) \tag{4}$$

Note that a_i has the character of a vector potential (connection) for an Abelian gauge theory, in that modifying $a_i(\xi)$ by a total derivative $a_i \to a_i + \frac{\partial}{\partial \xi^i}\theta$ does not affect dynamics, since the Lagrangian changes by a total time-derivative. Observe further that when a Hamiltonian is defined by the usual Legendre transform, velocities are absent from the combination $\frac{\partial L}{\partial \dot{\xi}^i}\dot{\xi}^i - L$, since L is first order in $\dot{\xi}^i$, and V may be identified with the Hamiltonian.

$$H = \frac{\partial L}{\partial \dot{\xi}^i}\dot{\xi}^i - L = V \tag{5}$$

Thus the Lagrangian in (4) may be presented as

$$L = a_i(\xi)\dot{\xi}^i - H(\xi) \tag{6}$$

and the first term on the right side defines the "canonical one-form" $a_i(\xi)\,d\xi^i \equiv a(\xi)$.

To introduce our method in its simplest realization, we begin with a special case, which in fact will be shown to be quite representative: instead of dealing with a general $a_i(\xi)$, we take it to be linear in ξ^i.

$$a_i(\xi) = \frac{1}{2}\xi^j \omega_{ji} \tag{7}$$

The constant matrix ω_{ij} is anti-symmetric, since any symmetric part merely contributes an irrelevant total time-derivative to L and can be dropped. The Euler-Lagrange equation that follows from (6) and (7) is

$$\omega_{ij}\dot{\xi}^j = \frac{\partial}{\partial\xi^i} H(\xi) \tag{8}$$

The development now goes to two cases. The first case holds when the anti-symmetric matrix ω_{ij} possesses an inverse, denoted by ω^{ij}, in which case ω_{ij} must be even-dimensional, *i.e.* the range N of $\{i, j, \ldots\}$ is $2n = N$. It follows from (7) that ξ^i satisfies the evolution equation

$$\dot{\xi}^i = \omega^{ij}\frac{\partial}{\partial\xi^j} H(\xi) \tag{9}$$

and *there are no constraints*. Constraints are present only in the second case, when ω_{ij} has no inverse, and as a consequence possesses N' zero modes $z_a^i,\ a = 1, \ldots, N'$. The system is then constrained by N' equations in which no time-derivatives appear.

$$z_a^i \frac{\partial}{\partial\xi^i} H(\xi) = 0 \tag{10}$$

On the space orthogonal to that spanned by the $\{z_a\}$, ω_{ij} possesses an even-dimensional ($= 2n$) inverse, so in this case $N = 2n + N'$.

For the moment we shall assume that ω_{ij} *does* possess an inverse and that there are no constraints. The second, constrained case will be dealt with later.

With the linear form for $a_i(\xi)$ and in the absence of constraints all dynamical equations are contained in (9). Brackets are defined so as to reproduce (9) by commutation with the Hamiltonian.

$$\begin{aligned}\dot{\xi}^i &= \omega^{ij}\frac{\partial}{\partial\xi^j} H(\xi) = i\left[H(\xi), \xi^i\right] \\ &= i\left[\xi^j, \xi^i\right]\frac{\partial}{\partial\xi^j} H(\xi)\end{aligned}$$

This implies that we should take

$$\left[\xi^i, \xi^j\right] = i\,\omega^{ij} \tag{11a}$$

or for general functions of ξ

$$[A(\xi), B(\xi)] = i\frac{\partial A(\xi)}{\partial\xi^i}\,\omega^{ij}\,\frac{\partial B(\xi)}{\partial\xi^j} \tag{11b}$$

It is reassuring to verify that a conventional dynamical model, when presented in the form (3), is a special case of the present theory with ξ^i comprising the two-component quantity $\binom{p}{q}$ and ω_{ij} the anti-symmetric 2×2 matrix ϵ_{ij}, $\epsilon_{12} = 1$. Eq. (11b) then implies $\lceil q, p \rfloor = i$.

Next let us turn to the more general case with $a_i(\xi)$ an arbitrary function of ξ^i, not depending explicitly on time. The Euler-Lagrange equation for (6) is

$$f_{ij}(\xi)\dot{\xi}^j = \frac{\partial}{\partial \xi^i} H(\xi) \tag{12}$$

where

$$f_{ij}(\xi) = \frac{\partial}{\partial \xi^i} a_j(\xi) - \frac{\partial}{\partial \xi^j} a_i(\xi) \tag{13}$$

f_{ij} behaves as a gauge invariant (Abelian) field strength (curvature) constructed from the gauge-variant potential (connection). It is called the "symplectic two-form," $\frac{1}{2} f_{ij}(\xi)\, d\xi^i\, d\xi^j = f(\xi)$; evidently it is exact: $f = da$, and therefore closed: $df = 0$. In the non-singular, unconstrained situation the anti-symmetric $N \times N$ matrix f_{ij} has the matrix inverse f^{ij}, hence $N = 2n$, and (12) implies

$$\dot{\xi}^i = f^{ij} \frac{\partial}{\partial \xi^j} H(\xi) \tag{14}$$

This evolution equation follows upon commutation with H provided the basic bracket is taken as

$$\left[\xi^i,\ \xi^j\right] = i f^{ij}(\xi) \tag{15}$$

The Bianchi identity satisfied by f_{ij} ensures that (15) obeys the Jacobi identity.

The result (15) and its special case (11b) can also be derived by an alternative, physically motivated argument. Consider a massive particle, in any number of dimensions, moving in an external electromagnetic field, described by the vector potential $a_i(\xi)$ and scalar potential $V(\xi)$. The Lagrangian and Hamiltonian are expressions familiar from the theory of the Lorentz force,

$$L = \frac{1}{2} m \dot{\xi}^i \dot{\xi}^i + a_i(\xi)\dot{\xi}^i - V(\xi) \tag{16a}$$

$$H = \frac{1}{2m} \left(p_i - a_i(\xi)\right)^2 + V(\xi) \tag{16b}$$

with p_i conjugate to ξ^i. It is seen that (4), (5) and (6) correspond to the $m \to 0$ limit of (16a) and (16b). Owing to the $0(m^{-1})$ kinetic term in (16b), the limit of vanishing mass can only be taken if $p_i - a_i(\xi) \equiv m\dot{\xi}^i$ is constrained to vanish. Adopting for the moment the Dirac procedure, we recognize that vanishing of $m\dot{\xi}^i$ is a second class constraint, since the constraints do not commute,

$$\begin{aligned}\left[m\dot{\xi}^i,\ m\dot{\xi}^j\right] &= [p_i - a_i(\xi),\ p_j - a_j(\xi)] \\ &= i\, f_{ij}(\xi) \neq 0 \end{aligned} \tag{17}$$

and computing the Dirac bracket $[\xi^i, \xi^j]$ regains (15).

In this way we see that what one would find by following Dirac is also gotten by our method, but we arrive at the goal much more quickly. Also the above discussion gives a physical setting for Lagrangians of the form (6): when dealing with a charged particle in an external magnetic field, in the strong field limit the Lorentz force term — the canonical one-form — dominates the kinetic term, which therefore may be dropped in first approximation. One is then left with quantum mechanical motion where the spatial coordinates fail to commute by terms of order of the inverse of the magnetic field. More specifically, with constant magnetic field B along the z-axis, energy levels of motion confined to the x-y plane form the well-known Landau bands. For strong fields, only the lowest band is relevant, and further effects of the additional potential $V(x, y)$ are approximately described by the "Peierls Substitution"[6]. This states that the low-lying energy eigenvalues are

$$E = \frac{B}{2m} + \epsilon_n \tag{18}$$

where $\frac{B}{2m}$ is the energy of the lowest Landau level in the absence of V, while the ϵ_n are eigenvalues of the operator $V(x, y)$ (properly ordered!) with

$$i[x,\, y] = \frac{1}{B} \tag{19}$$

Clearly the present considerations about quantizing first-order Lagrangians give a new derivation[7] of this ancient result from condensed matter physics.[6] [One may also verify (18) by forming mH from (16b) and computing ϵ_n perturbatively in m.[8]]

While the development starting with arbitrary $a_i(\xi)$ and unconstrained dynamics appears more general than that based on the linear, special case (7), the latter in fact includes the former. This is because by using Darboux's theorem one can show that an arbitrary vector potential [one-form $a_i\, d\xi^i$] whose associated field strength [two-form $d(a_i\, d\xi^i) = \frac{1}{2} f_{ij}\, d\xi^i\, d\xi^j$] is non-singular, in the sense that the matrix f_{ij} possesses an inverse, can be mapped by a coordinate transformation onto (7) with ω_{ij} non-singular. Thus apart from a gauge term, one can always present $a_i(\xi)$ as

$$a_i(\xi) = \frac{1}{2} Q^k(\xi)\, \omega_{k\ell}\, \frac{\partial Q^\ell(\xi)}{\partial \xi^i} \tag{20a}$$

correspondingly $f_{ij}(\xi)$ as

$$f_{ij}(\xi) = \frac{\partial Q^k(\xi)}{\partial \xi^i}\, \omega_{k\ell}\, \frac{\partial Q^\ell(\xi)}{\partial \xi^j} \tag{20b}$$

and in terms of new coordinates Q^i the curvature is ω_{ij} — a constant and non-singular matrix. Moreover, by a straightforward modification of the Gram-Schmidt argument a basis can be constructed such that the antisymmetric $N \times N$ matrix ω_{ij} takes the block-off-diagonal form

$$\omega_{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}_{ij} \tag{21}$$

where I is the n-dimensional unit matrix (N=2n). [With these procedures one can also handle the case when a_i is explicitly time-dependent — a transformation to constant ω_{ij} can still be made.] In the Appendix we present Darboux's theorem adopted for the present application, and we explicitly construct the coordinate transformation $Q^i(\xi)$. The coordinates in which the curvature two-form becomes (21) are of course the canonical coordinates and they can be renamed $p_i,\ q^i,\ i = 1, \ldots, n$.

We conclude the discussion of non-singular, first-order dynamics by recording the functional integral for the quantum theory. The action of (4) obviously is

$$I = \int a_i(\xi)\, d\xi^i - \int H(\xi)\, dt \tag{22}$$

and the path integral involves, as usual, the phase exponential of the action. The measure however is non-minimal; the correct prescription is

$$Z = \int \Pi_i \mathcal{D}\xi^i\ \det{}^{\frac{1}{2}} f_{jk}\ \exp i\, I \quad . \tag{23}$$

The $\det^{\frac{1}{2}} f_{ij}$ factor can be derived in a variety of ways: One may use Darboux's theorem to map the problem onto one with constant canonical curvature (21), where the measure is just the Liouville measure $\prod_{i=1}^{2n} \mathcal{D}\xi^i = \prod_{i=1}^{n} \mathcal{D}p_i\, \mathcal{D}q^i$, and the Jacobian of the transformation is seen from (20b) to be $\det^{\frac{1}{2}} f_{ij}$. Alternatively one may refer to our derivation based on Dirac's second class constraints, eqs. (16), (17), and recall that the functional integral in the presence of second class constraints involves the square root of the constraints' bracket[9]. By either argument, one arrives at (23), which also exhibits the essential nature of the requirement that f_{ij} be a non-singular matrix.

We now turn to the second, more complicated case, where there are constraints because f_{ij} is singular. It is evident from the Appendix that the Darboux construction may still be carried out for the non-singular projection of f_{ij}, which is devoid of the zero-modes (10). This results in the Lagrangian

$$L = \frac{1}{2}\, \xi^i\, \omega_{ij}\, \dot{\xi}^j - H(\xi,\, z) \tag{24}$$

Here ω_{ij} may still be taken in the canonical form (21), but now in the Hamiltonian there are N' additional coordinates, denoted by $z_a, a = 1, \ldots, N'$, arising from the N' zero modes of f_{ij} and leading to N' constraint equations.

$$\frac{\partial}{\partial z_a}\, H(\xi, z) = 0 \tag{25}$$

This is the form that (10) takes in the canonical coordinates achieved by Darboux's theorem. The constrained nature of the z_a variables is evident: they do not occur in the canonical one-form $\frac{1}{2}\, \xi^i\, \omega_{ij}\, \dot{\xi}^j\, dt$ and there is no time-development for them.

In the next step, we examine the constraint equations (25) and recognize that for the z_a occurring *non-linearly* in $H(\xi, z)$ one can solve (25) for the z_a. [More precisely, this needs $\det \frac{\partial^2 H(\xi,z)}{\partial z_a\, \partial z_b} \neq 0$.] On the other hand, when $H(\xi, z)$ contains a constrained z_a variable *linearly*, Eq. (25) does not permit an evaluation of the corresponding z_a, because (25) in that case is a relation among the ξ^i, with z_a absent from the equation. Therefore using (25), we evaluate as many z_a's as possible, in terms of ξ^i's and other z_a's, and leave for further analysis the linearly occurring z_a's. Note that this step does not affect the canonical one-form in the Lagrangian.

Upon evaluation and elimination of as many z_a's as possible, we are left with a Lagrangian in the form

$$L = \frac{1}{2}\xi^i \omega_{ij} \dot{\xi}^j - H(\xi) - \lambda_k \Phi^k(\xi) \tag{26}$$

where the last term arises from the remaining, linearly occurring z_a's, now renamed λ_k, and the only true constraints in the model are the Φ^k, which enter multiplied by Lagrange multipliers λ_k. To incorporate the constraints, it is not necessary to classify them into first class or second class. Rather we solve them, by satisfying the equations

$$\Phi^k(\xi) = 0 \tag{27}$$

which evidently give relations among the ξ^i — evaluating some in terms of others. This procedure obviously eliminates the last term in (26) and it reduces the number of ξ^i's below the $2n$ that are present in (26); also it replaces the diagonal canonical one-form by the expression $\bar{a}_i(\xi)\, d\xi^i$, where i ranges over the reduced set, and $\bar{a}_i$ is a non-linear function obtained by inserting the solutions to (27) into (26).

The Darboux procedure must now be repeated: the new canonical one-form $\bar{a}_i(\xi)\, d\xi^i$, which could be singular, is brought again to diagonal form, possibly leading to constraint equations, which must be solved. Eventually one hopes that the iterations terminate and one is left with a completely reduced, unconstrained and canonical system.

Of course there may be the technical obstacles to carrying out the above steps: solving the constraints may prove too difficult, constructing the Darboux transformation to canonical coordinates may not be possible. One can then revert to the Dirac method, with its first and second class constraints, and corresponding modifications of brackets, subsidiary conditions on states, and non-minimal measure factors in functional integrals.

I conclude my presentation by exhibiting our method in action for electromagnetism coupled to matter, which for simplicity I take to be Dirac fields ψ, since their Lagrangian is already first order. Also I include a gauge non-invariant mass term for the photon, to illustrate various examples of constraints. The electromagnetic Lagrangian in first-order form reads

$$L = \int d\mathbf{r} \left\{ -\mathbf{E} \cdot \dot{\mathbf{A}}^i + i\psi^* \dot{\psi} - \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2 + \mu^2 \mathbf{A}^2\right) \right\}$$

$$- H_M \left((\boldsymbol{\nabla} - i\mathbf{A})\psi \right)$$
$$- \int d\mathbf{r} \left\{ A_0 \left(\rho - \boldsymbol{\nabla} \cdot \mathbf{E} \right) - \frac{\mu^2}{2} A_0^2 \right\} \tag{28}$$

Here $\mathbf{A}$ is the vector potential with $\mathbf{B} \equiv \boldsymbol{\nabla} \times \mathbf{A}$, A_0 the scalar potential that is absent from the symplectic term, μ the photon mass. The matter Hamiltonian is not specified beyond an indication that coupling to $\mathbf{A}$ is through the covariant derivative, while $\rho = \psi^* \psi$. The Lagrangian is in the form (24); when μ is non-zero the constrained variable A_0 enters quadratically and

$$\frac{\delta H}{\delta A_0(\mathbf{r})} = 0 \tag{29}$$

leads to the evaluation of A_0

$$A_0 = \frac{1}{\mu^2} \left(\rho - \boldsymbol{\nabla} \cdot \mathbf{E} \right) \tag{30}$$

so that the unconstrained Lagrangian becomes

$$L = \int d\mathbf{r} \left\{ -\mathbf{E} \cdot \dot{\mathbf{A}} + i\psi^* \dot{\psi} - \frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2 + \mu^2 \mathbf{A}^2 + \frac{1}{\mu^2} \left(\rho - \boldsymbol{\nabla} \cdot \mathbf{E} \right)^2 \right) \right\} - H_M \left((\boldsymbol{\nabla} - i\mathbf{A})\, \psi \right) \tag{31}$$

The canonical pairs are identified as $(-\mathbf{E}, \mathbf{A})$ and $(i\psi^*, \psi)$. In the absence of a photon mass, the Lagrangian (28) is of the form (26), with one Lagrange multiplier $\lambda = A_0$. Eq. (29) then leads to the Gauss law constraint.

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho \tag{32}$$

To solve the constraint, we decompose both $\mathbf{E}$ and $\mathbf{A}$ into transverse and longitudinal parts,

$$\mathbf{E} = \mathbf{E}_T + \frac{\boldsymbol{\nabla}}{\sqrt{-\nabla^2}}\, E \tag{33}$$
$$\mathbf{A} = \mathbf{A}_T + \frac{\boldsymbol{\nabla}}{\sqrt{-\nabla^2}}\, A \tag{34}$$
$$\boldsymbol{\nabla} \cdot \mathbf{E}_T = \boldsymbol{\nabla} \cdot \mathbf{A}_T = 0$$

and (32) implies $E = \frac{-1}{\sqrt{-\nabla^2}}\, \rho$. Inserting this into (28) at $\mu^2 = 0$, we are left with

$$L = \int d\mathbf{r} \left\{ -\mathbf{E}_T \cdot \dot{\mathbf{A}}_T + \rho \frac{1}{\sqrt{-\nabla^2}} \dot{A} + i\psi^* \dot{\psi} - \frac{1}{2} \left(\mathbf{E}_T^2 + \mathbf{B}^2 - \rho \frac{1}{\nabla^2} \rho \right) \right\}$$
$$- H_M \left(\left(\boldsymbol{\nabla} - i\mathbf{A}_T - i\frac{\boldsymbol{\nabla}}{\sqrt{-\nabla^2}} A \right) \psi \right) \tag{35a}$$

While the constraint has been eliminated, the canonical one-form in (35a) is not diagonal. The Darboux transformation that is now performed replaces ψ by $\left(\exp i \frac{1}{\sqrt{-\nabla^2}} A\right) \psi$. This has the effect of canceling $\rho \frac{1}{\sqrt{-\nabla^2}} \dot{A}$ against a contribution coming from $i\psi^* \dot{\psi}$ and eliminating A from the Hamiltonian (since $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}_T$). We are thus left with the Coulomb-gauge Lagrangian

$$L = \int d\mathbf{r} \left\{ -\mathbf{E}_T \cdot \dot{\mathbf{A}}_T + i\psi^* \dot{\psi} - \frac{1}{2} \left(\mathbf{E}_T^2 + \mathbf{B}^2 - \rho \frac{1}{\nabla^2} \rho \right) \right\} - H_M \left((\boldsymbol{\nabla} - i\mathbf{A}_T) \psi \right) \tag{35b}$$

without ever selecting the Coulomb gauge! The canonical pairs are $(-\mathbf{E}_T, \mathbf{A}_T)$ and $(i\psi^*, \psi)$.

We recall that the Dirac approach would introduce a canonical momentum Π_0 conjugate to A_0 and constrained to vanish. The constraints (30) or (32) would then emerge as secondary constraints, which must hold so that $[H, \Pi_0]$ vanish. Finally a distinction would be made between the $\mu \neq 0$ and $\mu = 0$ theories: in the former the constraint is second class, in the latter it is first class.[9] None of these considerations are necessary for successful quantization.

Our method also quantizes very efficiently Chern-Simons theories, with or without a conventional kinetic term for the gauge field[10] [indeed the phase space reductive limit of taking the kinetic term to zero, as in (16), (17) above, can be clearly described[11]] as well as gravity theories in first order form, be they the Einstein model[12] or the recently discussed gravitational gauge theories in lower dimensions[13].

Finally, we record a first order Lagrangian L for Maxwell theory with external, conserved sources $(\rho, \mathbf{j})$, $\dot{\rho} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0$, which depends only on field strengths ($\mathbf{E}$, $\mathbf{B}$) (rather than potentials) and is self-dual in the absence of sources.

$$\begin{aligned} L = \int d\mathbf{r}\, d\mathbf{r}' \, \left(\dot{E}^i(\mathbf{r}) + j^i(\mathbf{r}) \right) \, \omega_{ij} \left(\mathbf{r} - \mathbf{r}' \right) B^j(\mathbf{r}') \\ - \frac{1}{2} \int d\mathbf{r} \, \left(\mathbf{E}^2 + \mathbf{B}^2 \right) - \int d\mathbf{r} \left(\lambda_1 (\rho - \boldsymbol{\nabla} \cdot \mathbf{E}) + \lambda_2 \boldsymbol{\nabla} \cdot \mathbf{B} \right) \end{aligned} \tag{36}$$

$$\omega_{ij}(\mathbf{r}) \equiv \epsilon^{ijk} \, \frac{\partial_k}{\nabla^2} \, \delta(\mathbf{r}) = \frac{1}{4\pi} \epsilon^{ijk} \, \frac{r^k}{r^3} \tag{37}$$

Varying the $\mathbf{E}$ and $\mathbf{B}$ fields as well as the two Lagrange multipliers $\lambda_{1,2}$ gives the eight Maxwell equations. The duality transformation $\mathbf{E} \to \mathbf{B}$, $\mathbf{B} \to -\mathbf{E}$, supplemented by $\lambda_1 \to -\lambda_2$, $\lambda_2 \to \lambda_1$ changes the Lagrangian by a total time derivative, when there are no sources. The canonical one-form is spatially non-local, owing to the presence of ω_{ij}, which has the inverse

$$\omega^{ij}(\mathbf{r}) = -\epsilon^{ijk} \, \partial_k \, \delta(\mathbf{r}) \tag{38}$$

when restricted to transverse fields — these are the only unconstrained degrees of freedom in (36). It then follows that the non-vanishing commutator is the familiar formula.

$$\Big[E^i_T(\mathbf{r}),\ B^j_T(\mathbf{r}')\Big] = -i\,\epsilon^{ijk}\,\partial_k\,\delta(\mathbf{r}-\mathbf{r}') \tag{39}$$

This self-dual presentation of electrodynamics is similar to formulations of self-dual fields on a line[5] and on a plane[10].

Appendix

Darboux's Theorem

We give a constructive derivation of Darboux's Theorem. Specifically we show that subject to regularity requirements stated below, any vector potential (connection one-form) $a_i(\xi)$ may be presented, apart from a gauge transformation, as

$$a_i(\xi) = \frac{1}{2} Q^m(\xi)\,\omega_{mn}\,\frac{\partial Q^n(\xi)}{\partial \xi^i} \tag{A.1}$$

and correspondingly the field strength $f_{ij}(\xi)$ (curvature two-form) as

$$f_{ij}(\xi) = \frac{\partial Q^m(\xi)}{\partial \xi^i}\,\omega_{mn}\,\frac{\partial Q^n(\xi)}{\partial \xi^j} \tag{A.2}$$

with ω_{mn} constant and anti-symmetric. The proof also gives a procedure for finding $Q^m(\xi)$. It is then evident that a coordinate transformation from ξ to Q renders f_{ij} constant and a further adjustment of the basis puts ω_{ij} in the canonical form (21).

We consider a continuously evolving transformation $Q^m(\xi;\tau)$, to be specified later, with the property that at $\tau = 0$, it is the identity transformation

$$Q^m(\xi;\,0) = \xi^m \tag{A.3a}$$

and at $\tau = 1$, it arrives at the desired $Q^m(\xi)$, (which will be explicitly constructed).

$$Q^m(\xi;\,1) = Q^m(\xi) \tag{A.3b}$$

$Q^m(\xi;\,\tau)$ is generated by $v^m(\xi;\,\tau)$, in the sense that

$$\frac{\partial}{\partial\tau}\,Q^m(\xi;\,\tau) = v^m\left(Q(\xi;\,\tau);\,\tau\right) \tag{A.4}$$

Note, that v^m depends explicitly on τ. Also we need to define the transform by $Q^m(\xi;\,\tau)$ of quantities relevant to the argument: connection one-form, curvature two-form *etc.* The definition is standard: the transform, denoted by T_Q, acts by

$$T_Q\;a_i(\xi) = a_m(Q)\,\frac{\partial Q^m}{\partial\xi^i} \tag{A.5a}$$

$$T_Q\;f_{ij}(\xi) = f_{mn}(Q)\,\frac{\partial Q^m}{\partial\xi^i}\,\frac{\partial Q^n}{\partial\xi^j} \tag{A.5b}$$

To give the construction, we consider the given $a_i(\xi)$ to be embedded in a one-parameter family $a_i(\xi;\,\tau)$, such that at $\tau = 0$ we have $a_i(\xi)$ and at $\tau = 1$ we have $\frac{1}{2}\,\xi^m\,\omega_{mi}$, where ω_{mi} is constant and anti-symmetric.

$$a_i(\xi;\,0) = a_i(\xi) \tag{A.6a}$$

$$a_i(\xi;\,1) = \frac{1}{2}\,\xi^m\,\omega_{mi} \tag{A.6b}$$

It is then true that

$$\frac{d}{d\tau}\left(T_Q\,a_i(\xi;\,\tau)\right) = T_Q\left(L_v\,a_i(\xi;\,\tau) + \frac{\partial}{\partial\tau}\,a_i(\xi;\,\tau)\right) \tag{A.7}$$

where L_v is the Lie derivative, with respect to the vector v^m that generates the transformation, see (A.4). Eq. (A.7) is straightforwardly verified by differentiating with respect to τ, and recalling that both the transformation and a_i are τ-dependent. Next we use the identity[14]

$$L_v\,a_i = v^n\,f_{ni} + \partial_i(v^n a_n) \tag{A.8}$$

and observe that when the generator is set equal to

$$v^n(\xi;\,\tau) = -f^{ni}(\xi;\,\tau)\frac{\partial}{\partial\tau}\,a_i(\xi;\,\tau) \tag{A.9}$$

Eq. (A.7) leaves

$$\frac{d}{d\tau}\left(T_Q\,a_i\right) = T_Q\left(\partial_i(v^n a_n)\right) \tag{A.10}$$

Thus $\frac{d}{d\tau}(T_Q a_i)$ is a gauge transformation, so that $T_Q a_i$ at $\tau = 0$, *i.e.* $a_i(\xi)$, differs from its value at $\tau = 1$, *i.e.* $\frac{1}{2} Q^m(\xi)\, \omega_{mn} \frac{\partial Q^n(\xi)}{\partial \xi^i}$, by a gauge transformation. This is the desired result, and moreover $Q^m(\xi; \tau)$ and $Q^m(\xi) \equiv Q^m(\xi; 1)$ are here explicitly constructed from the algebraic definition (A.9) for v^n [once an interpolating $a_i(\xi; \tau)$ is chosen], and integration of (A.4) (the latter task need not be easy).

Clearly (A.9) requires that $f_{ij}(\xi; \tau)$ possesses the inverse $f^{ij}(\xi; \tau)$; hence both the starting and ending forms, $f_{ij}(\xi)$ and ω_{ij}, must be non-singular. Also $f_{ij}(\xi; \tau)$ must remain non-singular for all intermediate τ. In fact this is not a restrictive requirement, because one may always choose ω_{ij} to be the value of $f_{ij}(\xi)$ at some point $\xi = \xi_0$, and then by change of basis transform ω_{ij} to any desired form.

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