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The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant

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Abstract. Liouville theory is shown to describe the asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant. This is because (i) Chern–Simons theory with a gauge group $SL(2, R) \times SL(2, R)$ on a spacetime with a cylindrical boundary is equivalent to the non-chiral $SL(2, R)$ WZW model; and (ii) the anti-de Sitter boundary conditions implement the constraints that reduce the WZW model to the Liouville theory.

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1. Introduction

Three-dimensional gravity theories have attracted considerable attention in the past fifteen years in the hope of getting a better understanding of the intricacies of their four-dimensional parents (see [1–3] for a recent review with an extensive list of references). In particular, the asymptotic structure of 3D gravity and 3D supergravity has been investigated [4–6], with the following conclusions. In the case of a vanishing cosmological constant, the asymptotic behaviour of the gravitational field is quite constrained and does not allow one to define naturally translations and supersymmetries at spatial infinity. (This property has been used recently to explain how one could break supersymmetry without generating a cosmological constant [7].) By contrast, the asymptotic structure of 3D gravity with a negative cosmological constant $\Lambda < 0$, is much richer§.

Indeed, in that case the asymptotic symmetry group turns out to be the group of conformal transformations in two dimensions, generated by the infinite-dimensional Virasoro algebra. The appearance of this conformal symmetry can be understood either in terms of the Penrose description of infinity by means of a conformal compactification, where infinity appears as a timelike cylinder ($\Lambda < 0$) and the asymptotic symmetry group is the group of its conformal symmetries [8]; or in terms of the Hamiltonian formulation where the canonical generators of the transformations preserving the boundary conditions are shown to close according to the conformal algebra [6].

The presence of the infinite-dimensional conformal group as asymptotic symmetry group suggests strongly that the asymptotic dynamics of the gravitational field in three dimensions, with a negative cosmological constant, is described by a two-dimensional conformal field

§ The assumption made in footnote 11 of [4] that the contraction from $\Lambda < 0$ to $\Lambda = 0$ can be done smoothly is thus incorrect.

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theory. The purpose of this paper is to show that the conformal field theory in question is the Liouville theory†.

Our starting point is the crucial observation made in [11, 12] that three-dimensional Einstein gravity with $\Lambda < 0$ can be reformulated as a Chern–Simons gauge theory with gauge group $SL(2, R) \times SL(2, R)$ and action

$$S_{\mathrm{E}}[A, \tilde{A}] = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\tilde{A}] \,. \tag{1}$$

Here A (respectively $\tilde{A}$) is the gauge field associated with the first (respectively the second) $SL(2, R)$ factor and S_{CS} is the Chern–Simons action which in polar coordinates t, r, φ takes the form

$$S_{\mathrm{CS}} = \int \mathrm{d}t\, \mathrm{d}r\, \mathrm{d}\varphi\ \mathrm{tr}(\dot{A}_r A_\varphi - \dot{A}_\varphi A_r - A_0 F_{r\varphi}) \,. \tag{2}$$

The connections A and $\tilde{A}$ are related to the triad e and spin connection ω through $A = e+\omega$, $\tilde{A} = e - \omega$.

The surface at spatial infinity ($r = \infty$) of anti-de-Sitter space is a timelike cylinder with coordinates t, φ. We denote it by Σ_2. The boundary conditions on the metric given in [6] read, when translated in terms of the connection A and $\tilde{A}$,

$$A \sim \begin{bmatrix} \dfrac{\mathrm{d}r}{2r} & \mathrm{O}(1/r) \\ r\,\mathrm{d}x^+ & -\dfrac{\mathrm{d}r}{2r} \end{bmatrix}, \qquad \tilde{A} \sim \begin{bmatrix} -\dfrac{\mathrm{d}r}{2r} & r\,\mathrm{d}x^- \\ \mathrm{O}(1/r) & \dfrac{\mathrm{d}r}{2r} \end{bmatrix} \tag{3}$$

(to leading order). Here, $x^{\pm} = t \pm \varphi$. The boundary conditions express that the metric approaches asymptotically the anti-de Sitter space and are, in particular, such that the triad e is non-degenerate.

Two things should be emphasized about (3). (i) The lightlike components A_- of A and $\tilde{A}_+$ of $\tilde{A}$ are set equal to zero asymptotically. (ii) $A_+^{(-)}$ and $\tilde{A}_-^{(+)}$ are not functions of the variables t and φ to leading order in r. At the same time, $A_+^{(3)}$ and $\tilde{A}_-^{(3)}$ are set to vanish. Here, the indices in parentheses are Lie algebra indices. We shall examine, in turn, the respective implications of (i) and (ii). We start by showing that (i) reduces the Chern–Simons theory to the $SL(2, R)$ non-chiral Wess–Zumino–Witten model. To that end, we closely follow the work of [13], adapted to the boundary conditions at hand. In section 3 we show then that the implication of (ii) is to reduce this WZW model to the Liouville model.

2. From the Einstein action to the non-chiral $SL(2, R)$ Wess–Zumino–Witten model

The action (1) is not an extremum on-shell when $\tilde{A}_-$ and $\tilde{A}_+$ are required to vanish on the boundary. Rather, δS is then equal to the surface term $\delta[\int_{\Sigma_2} \mathrm{d}t\, \mathrm{d}\varphi\, \mathrm{tr}(A_\varphi^2 + \tilde{A}_\varphi^2)]$ on the surface Σ_2 at infinity. (We shall examine the terms that arise at t_1 and t_2 when discussing (ii).) In order to have $\delta S = 0$, one must therefore add to the action the surface term $-[\int_{\Sigma_2} \mathrm{d}t\, \mathrm{d}\varphi\, \mathrm{tr}(A_\varphi^2 + \tilde{A}_\varphi^2)]$, leading to the improved action

$$S[A, \tilde{A}] = S_{\mathrm{CS}}[A] - \int_{\Sigma_2} \mathrm{d}t\, \mathrm{d}\varphi\, \mathrm{tr}(A_\varphi^2) - S_{\mathrm{CS}}[\tilde{A}] - \int_{\Sigma_2} \mathrm{d}t\, \mathrm{d}\varphi\, \mathrm{tr}(\tilde{A}_\varphi^2). \tag{4}$$

The temporal component A_0 and $\tilde{A}_0$ of the vector potential appears as a Lagrange multiplier implementing the constraints $F_{r\varphi} = \tilde{F}_{r\varphi} = 0$. One can solve these constraints as

$$A_i = G_1^{-1}\partial_i G_1, \qquad \tilde{A}_i = G_2^{-1}\partial_i G_2 \tag{5}$$

† That there is a connection between three-dimensional gravity and Liouville theory is of course not new and has been discussed from a different perspective in [9, 10].

where G_1 and G_2 are given asymptotically by

$$G_1 \sim g_1(t,\varphi)\begin{bmatrix} \sqrt{r} & 0 \\ 0 & \dfrac{1}{\sqrt{r}} \end{bmatrix}, \qquad G_2 \sim g_2(t,\varphi)\begin{bmatrix} \dfrac{1}{\sqrt{r}} & 0 \\ 0 & \sqrt{r} \end{bmatrix} \tag{6}$$

and where $g_1(t,\varphi)$ and $g_2(t,\varphi)$ are arbitrary elements of $SL(2,R)$. With equation (6), the asymptotic behaviour of the radial components of A and $\tilde{A}$ coincide with the one of (3), while the tangential components behave as

$$A_\alpha \sim \begin{bmatrix} a_\alpha^{(3)} & \dfrac{a_\alpha^{(+)}}{r} \\ a_\alpha^{(-)}r & -a_\alpha^{(3)} \end{bmatrix}, \qquad \tilde{A}_\alpha \sim \begin{bmatrix} \tilde{a}_\alpha^{(3)} & \tilde{a}_\alpha^{(+)}r \\ \dfrac{\tilde{a}_\alpha^{(-)}}{r} & -\tilde{a}_\alpha^{(3)} \end{bmatrix} \tag{7}$$

where $a_\alpha = g_1^{-1}\partial_\alpha g_1$ and $\tilde{a}_\alpha = g_2^{-1}\partial_\alpha g_2$. The group elements $g_1(t,\varphi)$ and $g_2(t,\varphi)$ will be restricted below so that A_α and $\tilde{A}_\alpha$ fulfil the remaining boundary conditions (ii).

Strictly speaking (5) is valid only if the spatial sections have no hole. In general, one should allow for holonomies, which appear as additional 'zero-mode terms' in (5). Such additional terms are necessary to describe black holes in three dimensions which can be obtained from anti-de Sitter space by making appropriate identifications [14, 15]. Furthermore, there are then also additional inner boundaries with their own surface dynamics. The dynamics on a black hole horizon, has been treated in [16]. Since here we are only interested in the asymptotic dynamics of the gravitational field, we shall, however, drop the holonomies and ignore the inner surfaces. A full treatment will be given in [17].

Now, if one inserts (6) in the action (5), one gets

$$S[A,\tilde{A}] = S^{\rm R}_{\rm WZW}[g_1] - S^{\rm L}_{\rm WZW}[g_2] \tag{8}$$

where $S^{\rm R}_{\rm WZW}[g_1]$ and $S^{\rm L}_{\rm WZW}[g_2]$ are the two-dimensional chiral Wess–Zumino–Witten (WZW) actions [13, 18–21]. These first-order actions generalize the Abelian actions of [22] and describe a right-moving group element $g_1(x^+)$ and a left-moving group element $g_2(x_-)$, respectively,

$$S^{\rm R}_{\rm WZW}[g_1] = \int_{\Sigma_2} {\rm d}t\,{\rm d}\varphi\,{\rm tr}(\dot{g}_1 g_1' - (g_1')^2) + \Gamma[g_1] \tag{9}$$

$$S^{\rm L}_{\rm WZW}[g_2] = \int_{\Sigma_2} {\rm d}t\,{\rm d}\varphi\,{\rm tr}(\dot{g}_2 g_2' + (g_2')^2) + \Gamma[g_2] \tag{10}$$

where $\dot{g} = g^{-1}\frac{\partial}{\partial t}g$, $g' = g^{-1}\frac{\partial}{\partial\varphi}g$ and $\Gamma[g]$ is the usual three-dimensional part of the WZW action. As shown in [18–20], the actions (9) and (10) each lead to a single chiral Kac–Moody symmetry (of opposite chirality). One expects the sum (8) of the left and right chiral actions (9), (10) to be equivalent to the standard, non-chiral, WZW action [23] with dynamical variable $g = g_1^{-1}g_2$ since in that model the right-moving and left-moving sectors are indeed decoupled [23, 24]. This expectation turns out to be true.

One way to establish the equivalence is to rewrite the standard WZW action in Hamiltonian form, since (9) and (10) are linear and of first order in the time derivatives. We denote by Π_g the momentum conjugate to g and by u the function of g and Π_g which is equal to $\dot{g}$ when the equations of motion hold. One may take g and u as independent variables. The change of variables

$$g = g_1^{-1}g_2, \qquad u \equiv \dot{g}|_{\text{on-shell}} = -g_2^{-1}\frac{\partial}{\partial\varphi}g_1 g_1^{-1}g_2 - g_2^{-1}\frac{\partial}{\partial\varphi}g_2 \tag{11}$$

brings (8) to the standard WZW action in first-order form or, after elimination of the auxiliary field u, to the standard, non-chiral $SL(2,R)$ WZW action in second-order form,

$$S[A,\tilde{A}] = S^{\mathrm{WZW}}[g], \qquad S^{\mathrm{WZW}}[g] = \int_{\sigma_2} \mathrm{d}t\,\mathrm{d}\varphi\,(\mathrm{tr}(g_+g_-) - \Gamma[g]) \tag{12}$$

where $g_\pm \equiv g^{-1}\frac{\partial}{\partial x^\pm}g$. We omit the details here, leaving them for the complete treatment [17] where, in particular, the zero modes and holonomies will also be included. We simply note that the transformation (11) is the direct generalization of the transformation analysed in [18, 25] in the $U(1)$ case, which establishes the equivalence of the sum of left-moving chiral boson and a right-moving chiral boson to a massless Klein–Gordon field.

Thus, so far we have shown that the asymptotic dynamics of the gravitational field in three dimensions with $\Lambda < 0$ is described by the (non-chiral) $SL(2,R)$ WZW action. We have not yet, however, incorporated all the boundary conditions on the connection. This missing step is taken now.

3. From the WZW model to Liouville theory

The conditions that we have not taken into account at this stage are the conditions (ii) which read, in terms of the group element g

$$J_-^{(+)} \equiv (g^{-1}\partial_- g)^{(+)} = 1, \qquad \tilde{J}_+^{(-)} \equiv (\partial_+ g g^{-1})^{(-)} = 1 \tag{13}$$

and $J_+^{(3)} = 0$, $J_-^{(3)} = 0$. Since the J's are just the Kac–Moody currents of the WZW model, we recognise (13) as the conditions implementing the familiar Hamiltonian reduction of the WZW model to the Liouville theory. The conditions $J_+^{(3)} = J_-^{(3)} = 0$ appear as 'gauge condition'. This reduction has been discussed at length in the literature so that we do not need to recall the details here.

Let us simply point out the perhaps less familiar fact that the reduction can be carried out directly at the level of the action. As shown in [26], the WZW action reads, if one parametrizes g according to the Gauss decomposition

$$g = \begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \exp(\frac{1}{2}\phi) & 0 \\ 0 & \exp(-\frac{1}{2}\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}, \tag{14}$$

$$S^{\mathrm{WZW}}[g] = \int \mathrm{d}t\,\mathrm{d}\varphi(\tfrac{1}{2}\partial_+\phi\partial_-\phi + 2(\partial_- X)(\partial_+ Y)\exp(-\phi))\,. \tag{15}$$

Now the action (12) is defined on the cylinder $[t_1, t_2] \times S_1$ of finite height $t_2 - t_1$ and is stationary on the classical history provided one fixes ϕ, X and Y at the time boundary t_1 and t_2. However, since we want to implement the constraints (13), it is not ϕ, X and Y that we want to fix at the boundaries, but rather, ϕ and the momentum $\partial_- Y$ and $\partial_+ X$ conjugate to X and Y, since $J_+^{(+)} = \partial_- X\exp(-\phi)$ and $\tilde{J}_-^{(-)} = \partial_+ Y\exp(-\phi)$. Hence the constraints (13) cannot be simply plugged into (15). The action appropriate to the new set of boundary conditions differs from (15) by a boundary term at t_1 and t_2,

$$S^{\mathrm{WZW}}_{\mathrm{impr}}[g] = S^{\mathrm{WZW}}[g] - 2\oint \mathrm{d}\varphi(X\partial_+ Y + Y\partial_- X)\exp(-\phi)|_{t_1}^{t_2}\,. \tag{16}$$

With the 'improved' term, it is legitimate to insert the constraints (13) in the action (16). If one does so, one ends up with the Liouville action for ϕ

$$S[A,\tilde{A}] = S_{\mathrm{Liouville}}[\phi] = \int \mathrm{d}t\,\mathrm{d}\varphi\big(\tfrac{1}{2}\partial_+\phi\partial_-\phi + 2\exp(\phi)\big)\,. \tag{17}$$

Hence, we have established that the asymptotic dynamics of the gravitational field in three dimensions, with $\Lambda < 0$, is indeed described by the Liouville theory. As is well known, this theory is conformally invariant and possesses two sets of Virasoro generators L_n and $\tilde{L}_n$. These can be viewed as generating the residual Kac–Moody symmetries preserving the constraints and are the asymptotic generators found by a totally different approach in [6]. (See also [27, 28] for a related analysis of the surface terms in the Chern–Simons theories.)

Remark. one can actually substitute the constraints $J_+^{(+)} = \mu$ and $\tilde{J}_-^{(-)} = \nu$ in the action (16), provided one observes that the constants μ and ν are functionals of the fields and varies them accordingly in the action principle. A very similar situation occurs when one treats the cosmological constant as a dynamical variable. The subtleties of the variational principle are explained in that case in [29].

4. Conclusions

In this paper, we have completed the analysis of the asymptotic dynamics of 3D Einstein gravity with a negative cosmological constant. We have shown that the Virasoro symmetry generators found in [6] arise because the asymptotic dynamics is described by a conformally invariant theory, namely the Liouville theory.

The asymptotic reduction of the Einstein action—equivalent to $SL(2, R) \times SL(2, R)$ Chern–Simons action—to the Liouville action follows a two-step procedure. First, one imposes conditions of opposite chiralities on each $SL(2, R)$ factor, namely $A_- = 0$ and $\tilde{A}^+ = 0$. This leads to the sum of two chiral $SL(2, R)$ WZW actions of opposite chiralities or, what is the same, to the non chiral $SL(2, R)$ WZW action. Next one imposes the constraints on the Kac–Moody currents that reduce the $SL(2, R)$ WZW action to the Liouville theory. The two steps are precisely incorporated in the boundary conditions on the triads and connection expressing asymptotic approach to the anti-de Sitter space, and thus, they have a direct geometrical interpretation. Our analysis also exemplifies very clearly how the asymptotic dynamics is sensitive to the boundary conditions.

A further account of this work, with extension to supersymmetry (important for proving positivity of the energy theorems) will be reported elsewhere.

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